

# INVERSE ZERO-SUM PROBLEMS III

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## 1. INTRODUCTION

We continue the investigations started in [7, 13]. Let  $G = C_n \oplus C_n$  with  $n \geq 2$ . We say that  $G$  has Property **B** if every minimal zero-sum sequence  $S$  over  $G$  of length  $|S| = 2n - 1$  contains an element with multiplicity  $n - 1$ . The aim of the present paper is to prove the following two results.

**Theorem.** *Let  $G = C_{mn} \oplus C_{mn}$  with  $m, n \geq 3$  odd and  $mn > 9$ . If both  $C_m \oplus C_m$  and  $C_n \oplus C_n$  have Property **B**, then  $G$  has Property **B**.*

**Corollary.** *Let  $G = C_{n_1} \oplus C_{n_2}$  with  $1 < n_1 \mid n_2$ , and suppose that, for every prime divisor  $p$  of  $n_1$ , the group  $C_p \oplus C_p$  has Property **B**. Then  $C_{n_1} \oplus C_{n_1}$  has Property **B**, and a sequence  $S$  over  $G$  of length  $D(G) = n_1 + n_2 - 1$  is a minimal zero-sum sequence if and only if it has one of the following two forms:*

•

$$S = e_j^{\text{ord}(e_j)-1} \prod_{\nu=1}^{\text{ord}(e_k)} (x_\nu e_j + e_k), \quad \text{where}$$

$(e_1, e_2)$  is a basis of  $G$  with  $\text{ord}(e_i) = n_i$  for  $i \in \{1, 2\}$ ,  $\{j, k\} = \{1, 2\}$ ,  $x_1, \dots, x_{\text{ord}(e_k)} \in [0, \text{ord}(e_j) - 1]$ , and  $x_1 + \dots + x_{\text{ord}(e_k)} \equiv 1 \pmod{\text{ord}(e_j)}$ .

•

$$S = g_1^{s n_1 - 1} \prod_{\nu=1}^{n_2 + (1-s)n_1} (-x_\nu g_1 + g_2), \quad \text{where}$$

$\{g_1, g_2\}$  is a generating set of  $G$  with  $\text{ord}(g_2) = n_2$ ,  $x_1, \dots, x_{n_2 + (1-s)n_1} \in [0, n_1 - 1]$ ,  $x_1 + \dots + x_{n_2 + (1-s)n_1} = n_1 - 1$ ,  $s \in [1, n_2/n_1]$ , and either  $s = 1$  or  $n_1 g_1 = n_2 g_2$ .

Thus Property **B** is multiplicative, and if  $G = C_{n_1} \oplus C_{n_2}$  with  $1 < n_1 \mid n_2$  is a group of rank two, and for every prime divisor  $p$  of  $n_1$  the group  $C_p \oplus C_p$  has Property **B**, then the minimal zero-sum sequences of maximal length over  $G$  are explicitly characterized.

In Section 2, we fix our notation and gather the necessary tools (apart from former work on Property **B** and classical addition theorems, we use a confirmed conjecture of Y.ould Hamidoune, see Theorem 2.7). Section 3 contains some straightforward lemmas. The proof of the Theorem consists of two major parts: the first is given in Section 4 and the second, more involved one, is given in Section 5.

The Corollary is mainly based on the Theorem above, on former work of the authors [5], and on recent work by Wolfgang A. Schmid [13]. Its proof only needs a few lines and is given in Section 6.

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## 2. PRELIMINARIES

Our notation and terminology are consistent with [7] and [9]. We briefly gather some key notions and fix the notation concerning sequences over abelian groups. Let  $\mathbb{N}$  denote the set of positive integers and let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . For real numbers  $a, b \in \mathbb{R}$ , we set  $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ . Throughout, all abelian groups will be written additively. For  $n \in \mathbb{N}$ , let  $C_n$  denote a cyclic group with  $n$  elements. Let  $G$  be an abelian group.

Let  $A, B \subset G$  be nonempty subsets. Then  $A + B = \{a + b \mid a \in A, b \in B\}$  denotes their *sumset* and  $A - B = \{a - b \mid a \in A, b \in B\}$  their *difference set*. The *stabilizer* of  $A$  is defined as  $\text{Stab}(A) = \{g \in G \mid g + A = A\}$ , and  $A$  is called *periodic* if  $\text{Stab}(A) \neq \{0\}$ .

An  $s$ -tuple  $(e_1, \dots, e_s)$  of elements of  $G$  is said to be *independent* if  $e_i \neq 0$  for all  $i \in [1, s]$  and, for every  $s$ -tuple  $(m_1, \dots, m_s) \in \mathbb{Z}^s$ ,

$$m_1 e_1 + \dots + m_s e_s = 0 \quad \text{implies} \quad m_1 e_1 = \dots = m_s e_s = 0.$$

An  $s$ -tuple  $(e_1, \dots, e_s)$  of elements of  $G$  is called a *basis* if it is independent and  $G = \langle e_1 \rangle \oplus \dots \oplus \langle e_s \rangle$ .

Let  $G = C_n \oplus C_n$  with  $n \geq 2$ , and let  $(e_1, e_2)$  be a basis of  $G$ . An endomorphism  $\varphi: G \rightarrow G$  with

$$(\varphi(e_1), \varphi(e_2)) = (e_1, e_2) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{where} \quad a, b, c, d \in \mathbb{Z},$$

is an automorphism if and only if  $(\varphi(e_1), \varphi(e_2))$  is a basis, which is equivalent to  $\gcd(ad - bc, n) = 1$ . If  $f_1 \in G$  with  $\text{ord}(f_1) = n$ , then clearly there is an  $f_2 \in G$  such that  $(f_1, f_2)$  is a basis of  $G$ .

Let  $\mathcal{F}(G)$  be the free monoid with basis  $G$ . The elements of  $\mathcal{F}(G)$  are called *sequences* over  $G$ . We write sequences  $S \in \mathcal{F}(G)$  in the form

$$S = \prod_{g \in G} g^{\mathbf{v}_g(S)}, \quad \text{with} \quad \mathbf{v}_g(S) \in \mathbb{N}_0 \quad \text{for all} \quad g \in G.$$

We call  $\mathbf{v}_g(S)$  the *multiplicity* of  $g$  in  $S$ , and we say that  $S$  *contains*  $g$  if  $\mathbf{v}_g(S) > 0$ . A sequence  $S_1$  is called a *subsequence* of  $S$  if  $S_1 \mid S$  in  $\mathcal{F}(G)$  (equivalently,  $\mathbf{v}_g(S_1) \leq \mathbf{v}_g(S)$  for all  $g \in G$ ). Given two sequences  $S, T \in \mathcal{F}(G)$ , we denote by  $\gcd(S, T)$  the longest subsequence dividing both  $S$  and  $T$ . If a sequence  $S \in \mathcal{F}(G)$  is written in the form  $S = g_1 \dots g_l$ , we tacitly assume that  $l \in \mathbb{N}_0$  and  $g_1, \dots, g_l \in G$ .

For a sequence

$$S = g_1 \cdot \dots \cdot g_l = \prod_{g \in G} g^{\mathbf{v}_g(S)} \in \mathcal{F}(G),$$

we call

$$|S| = l = \sum_{g \in G} \mathbf{v}_g(S) \in \mathbb{N}_0 \quad \text{the length of } S,$$

$$\mathbf{h}(S) = \max\{\mathbf{v}_g(S) \mid g \in G\} \in [0, |S|]$$

the *maximum of the multiplicities* of  $S$ ,

$$\text{supp}(S) = \{g \in G \mid \mathbf{v}_g(S) > 0\} \subset G \quad \text{the support of } S,$$

$$\sigma(S) = \sum_{i=1}^l g_i = \sum_{g \in G} \mathbf{v}_g(S)g \in G \quad \text{the sum of } S,$$

$$\Sigma_k(S) = \left\{ \sum_{i \in I} g_i \mid I \subset [1, l] \text{ with } |I| = k \right\}$$

the set of  $k$ -term subsums of  $S$ , for all  $k \in \mathbb{N}$ ,

$$\Sigma_{\leq k}(S) = \bigcup_{j \in [1, k]} \Sigma_j(S), \quad \Sigma_{\geq k}(S) = \bigcup_{j \geq k} \Sigma_j(S),$$

and

$$\Sigma(S) = \Sigma_{\geq 1}(S) \text{ the set of (all) subsums of } S.$$

The sequence  $S$  is called

- *zero-sum free* if  $0 \notin \Sigma(S)$ ,
- a *zero-sum sequence* if  $\sigma(S) = 0$ ,
- a *minimal zero-sum sequence* if  $1 \neq S$ ,  $\sigma(S) = 0$ , and every  $S'|S$  with  $1 \leq |S'| < |S|$  is zero-sum free.

We denote by  $\mathcal{A}(G) \subset \mathcal{F}(G)$  the set of all minimal zero-sum sequences over  $G$ . Every map of abelian groups  $\varphi: G \rightarrow H$  extends to a homomorphism  $\varphi: \mathcal{F}(G) \rightarrow \mathcal{F}(H)$  where  $\varphi(S) = \varphi(g_1) \cdot \dots \cdot \varphi(g_l)$ . We say that  $\varphi$  is *constant* on  $S$  if  $\varphi(g_1) = \dots = \varphi(g_l)$ . If  $\varphi$  is a homomorphism, then  $\varphi(S)$  is a zero-sum sequence if and only if  $\sigma(S) \in \text{Ker}(\varphi)$ .

**Definition 2.1.** Let  $G$  be a finite abelian group with exponent  $n$ .

1. Let  $D(G)$  denote the smallest integer  $l \in \mathbb{N}$  such that every sequence  $S \in \mathcal{F}(G)$  of length  $|S| \geq l$  has a zero-sum subsequence. Equivalently, we have  $D(G) = \max\{|S| \mid S \in \mathcal{A}(G)\}$ , and  $D(G)$  is called the *Davenport constant* of  $G$ .
2. Let  $\eta(G)$  denote the smallest integer  $l \in \mathbb{N}$  such that every sequence  $S \in \mathcal{F}(G)$  of length  $|S| \geq l$  has a zero-sum subsequence  $T$  of length  $|T| \in [1, n]$ .
3. We say that  $G$  has **Property C** if every sequence  $S$  over  $G$  of length  $|S| = \eta(G) - 1$ , with no zero-sum subsequence of length in  $[1, n]$ , has the form  $S = T^{n-1}$  for some sequence  $T$  over  $G$ .

**Lemma 2.2.** Let  $G = C_{n_1} \oplus C_{n_2}$  with  $1 \leq n_1 \mid n_2$ .

1. We have  $D(G) = n_1 + n_2 - 1$  and  $\eta(G) = 2n_1 + n_2 - 2$ .
2. If  $n_1 = n_2$  and  $G$  has **Property B**, then  $G$  has **Property C**.

*Proof.* 1. See [9, Theorem 5.8.3].

2. See [5, Theorem 6.2] and [6, Theorem 6.7.2.(b)]. □

Results on  $\eta(G)$  for groups of higher rank may be found in recent work of C. Elsholtz et.al. ([4, 3]).

**Lemma 2.3.** Let  $G = C_n \oplus C_n$  with  $n \geq 2$ .

1. Then the following statements are equivalent:
  - (a) If  $S \in \mathcal{F}(G)$ ,  $|S| = 3n - 3$  and  $S$  has no zero-sum subsequence  $T$  of length  $|T| \geq n$ , then there exists some  $a \in G$  such that  $0^{n-1}a^{n-2} \mid S$ .
  - (b) If  $S \in \mathcal{F}(G)$  is zero-sum free and  $|S| = 2n - 2$ , then  $a^{n-2} \mid S$  for some  $a \in G$ .
  - (c) If  $S \in \mathcal{A}(G)$  and  $|S| = 2n - 1$ , then  $a^{n-1} \mid S$  for some  $a \in G$ .

- (d) If  $S \in \mathcal{A}(G)$  and  $|S| = 2n - 1$ , then there exists a basis  $(e_1, e_2)$  of  $G$  and integers  $x_1, \dots, x_n \in [0, n - 1]$ , with  $x_1 + \dots + x_n \equiv 1 \pmod{n}$ , such that

$$S = e_1^{n-1} \prod_{\nu=1}^n (x_\nu e_1 + e_2).$$

2. Let  $S \in \mathcal{A}(G)$  be of length  $|S| = 2n - 1$  and  $e_1 \in G$  with  $v_{e_1}(S) = n - 1$ . If  $(e_1, e'_2)$  is a basis of  $G$ , then there exist some  $b \in [0, n - 1]$  and  $a'_1, \dots, a'_n \in [0, n - 1]$ , with  $\gcd(b, n) = 1$  and  $\sum_{\nu=1}^n a'_\nu \equiv 1 \pmod{n}$ , such that

$$S = e_1^{n-1} \prod_{\nu=1}^n (a'_\nu e_1 + b e'_2).$$

3. If  $S \in \mathcal{A}(G)$  has length  $|S| = 2n - 1$ , then  $\text{ord}(g) = n$  for all  $g \in \text{supp}(S)$ .

*Proof.* 1. See [9, Theorem 5.8.7].

2. This follows easily from 1; for details see [5, Proposition 4.1].

3. See [9, Theorem 5.8.4]. □

The characterization in Lemma 2.3.1 gives rise to the following definition.

**Definition 2.4.** Let  $G = C_n \oplus C_n$  with  $n \geq 2$ .

1. Let  $\Upsilon(G)$  be the set of all  $S \in \mathcal{A}(G)$  for which there exists a basis  $(e_1, e_2)$  of  $G$  and integers  $x_1, \dots, x_n \in [0, n - 1]$ , with  $x_1 + \dots + x_n \equiv 1 \pmod{n}$ , such that  $S = e_1^{n-1} \prod_{\nu=1}^n (x_\nu e_1 + e_2)$ .
2. Let  $\Upsilon_u(G)$  be the set of those  $S \in \Upsilon(G)$  with a unique term of multiplicity  $n - 1$ , and let  $\Upsilon_{nu}(G) = \Upsilon(G) \setminus \Upsilon_u(G)$ .

Thus, by Lemma 2.3.1, a group  $G = C_n \oplus C_n$  with  $n \geq 2$  has Property **B** if and only if  $\mathcal{A}(G) = \Upsilon(G)$ .

**Lemma 2.5.** Let  $G = C_{mn} \oplus C_{mn}$  with  $m, n \geq 2$ , let  $S \in \mathcal{A}(G)$  be of length  $|S| = 2mn - 1$ , and let  $\varphi: G \rightarrow G$  denote the multiplication by  $m$  homomorphism.

1.  $\varphi(S)$  is not a product of  $2m$  zero-sum subsequences. Every zero-sum subsequence  $T$  of  $\varphi(S)$  of length  $|T| \in [1, n]$  has length  $n$ , and  $0 \notin \text{supp}(\varphi(S))$ .
2.  $S$  may be written in the form  $S = W_0 \cdot \dots \cdot W_{2m-2}$ , where  $W_0, \dots, W_{2m-2} \in \mathcal{F}(G)$  with  $|W_0| = 2n - 1$ ,  $|W_1| = \dots = |W_{2m-2}| = n$  and  $\sigma(W_0), \dots, \sigma(W_{2m-2}) \in \text{Ker}(\varphi)$ .

*Proof.* See [5, Lemma 3.14]. □

The following is the Erdős-Ginzburg-Ziv Theorem and the corresponding characterization of extremal sequences.

**Theorem 2.6.** Let  $G$  be a cyclic group of order  $n \geq 2$  and  $S \in \mathcal{F}(G)$ .

1. If  $|S| \geq 2n - 1$ , then  $0 \in \Sigma_n(S)$ .
2. If  $|S| = 2n - 2$  and  $0 \notin \Sigma_n(S)$ , then  $S = g^{n-1} h^{n-1}$  for some  $g, h \in G$  with  $\text{ord}(g - h) = n$ .

*Proof.* 1. See [9, Corollary 5.7.5] or [12, Theorem 2.5].

2. See [2, Lemma 4] for one of the original proofs, and [8, Section 7.A].  $\square$

The following result was a conjecture of Y.ould Hamidoune [11] confirmed in [10, Theorem 1].

**Theorem 2.7.** *Let  $G$  be a finite abelian group,  $S \in \mathcal{F}(G)$  of length  $|S| \geq |G| + 1$ , and  $k \in \mathbb{N}$  with  $k \leq |\text{supp}(S)|$ . If  $h(S) \leq |G| - k + 2$  and  $0 \notin \Sigma_{|G|}(S)$ , then  $|\Sigma_{|G|}(S)| \geq |S| - |G| + k - 1$ .*

### 3. PREPARATORY RESULTS.

We first prove several lemmas determining in what ways a sequence  $S \in \Upsilon(C_m \oplus C_m)$ , where  $m \geq 4$ , can be slightly perturbed and still remain in  $\Upsilon(C_m \oplus C_m)$ . These will later be heavily used in Section 5, always in the setting where  $K = \text{Ker}(\varphi)$  and  $\varphi: G \rightarrow G$  is the multiplication by  $m$  map.

**Lemma 3.1.** *Let  $K = C_m \oplus C_m$  with  $m \geq 4$ , let  $g \in K$ , and let  $S = f_1^{m-1} \prod_{\nu=1}^m (x_\nu f_1 + f_2) \in \Upsilon_u(K)$  with  $x_1, \dots, x_m \in \mathbb{Z}$ .*

1. *If  $S' = f_1^{-2} S(f_1 + g)(f_1 - g) \in \Upsilon(K)$ , then  $g = 0$  and hence  $S = S'$ .*
2. *If  $S' = f_1^{-1} (x_j f_1 + f_2)^{-1} S(f_1 + g)(x_j f_1 + f_2 - g) \in \Upsilon(K)$ , then  $g \in \{0, (x_j - 1)f_1 + f_2\}$  and hence  $S = S'$ .*
3. *If  $S' = (x_j f_1 + f_2)^{-1} (x_k f_1 + f_2)^{-1} S(x_j f_1 + f_2 + g)(x_k f_1 + f_2 - g) \in \Upsilon(K)$  with  $j, k \in [1, m]$  distinct, then  $g \in \langle f_1 \rangle$ .*

*Proof.* 1. Assume to the contrary that  $g \neq 0$  and thus  $S \neq S'$ . Then  $v_{f_1}(S') < m - 1$  and, since  $S \in \Upsilon_u(K)$ , it follows that there is some  $j \in [1, m]$  such that  $(x_j f_1 + f_2)^{m-1} | S'$ ,  $(x_j f_1 + f_2)^{m-3} | S$ , and  $x_j f_1 + f_2 = f_1 + g$ . If we set  $f'_2 = x_j f_1 + f_2$ , then  $S = f_1^{m-1} \prod_{\nu=1}^m ((x_\nu - x_j) f_1 + f'_2)$ , and thus we may assume that  $f_2 = f'_2$ . Then  $f_2 = f_1 + g$  and  $f_1 - g = f_2 - 2g = 2f_1 - f_2$ . Since  $m \geq 4$ , it follows that  $f_1 | S'$ . Since  $S' \in \Upsilon(K)$ ,  $f_2^{m-1} | S'$  and  $f_1, 2f_1 - f_2 \in \text{supp}(S') \setminus \{f_2\}$ , it follows that  $(2f_1 - f_2) - f_1 = f_1 - f_2 \in \langle f_2 \rangle$ , a contradiction.

2. After renumbering, we may suppose that  $j = n$ . If  $f_1^{m-1} | S'$  then  $f_1 + g = f_1$  or  $x_n f_1 + f_2 - g = f_1$ , and  $S' = S$ . Otherwise,  $f_1^{m-1} \nmid S'$  and we shall derive a contradiction. Observe that we cannot have  $f_1 + g = x_n f_1 + f_2 - g = x_j f_1 + f_2$ . Thus, since  $S' \in \Upsilon(K)$  and  $S \in \Upsilon_u(K)$ , it follows that (after renumbering again if necessary) either

$$S' = f_1^{m-2} (x f_1 + f_2)^{m-1} (x_n f_1 + f_2 - g)(x_{n-1} f_1 + f_2) \quad \text{with} \quad f_1 + g = x f_1 + f_2,$$

or

$$S' = f_1^{m-2} (x f_1 + f_2)^{m-1} (f_1 + g)(x_{n-1} f_1 + f_2) \quad \text{with} \quad x_n f_1 + f_2 - g = x f_1 + f_2.$$

In the first case, we have  $(x_n f_1 + f_2 - g) = (x_n - x + 1) f_1$  and hence  $f_1^{m-2} ((x_n - x + 1) f_1) | S'$ . However, since  $(x_n - x + 1) f_1 = (x_n f_1 + f_2 - g) \neq f_1$ , it follows that  $f_1^{m-2} ((x_n - x + 1) f_1)$  is not zero-sum free, a contradiction. In the second case, one can derive a contradiction similarly.

3. Since  $m \geq 3$ ,  $f_1^{m-1} | S'$  and  $S' \in \Upsilon(K)$ , it follows that  $(x_j f_1 + f_2 + g) - (x_l f_1 + f_2) \in \langle f_1 \rangle$ , where  $l \neq j, k$ , and hence  $g \in \langle f_1 \rangle$ .  $\square$

**Lemma 3.2.** *Let  $K = C_m \oplus C_m$  with  $m \geq 4$ ,  $g \in K$  and  $S = f_1^{m-1}f_2^{m-1}(f_1 + f_2) \in \Upsilon_{nu}(K)$ .*

1. *If  $S' = f_1^{-2}S(f_1 + g)(f_1 - g) \in \Upsilon(K)$ , then  $g \in \langle f_2 \rangle$ .*
2. *If  $S' = f_2^{-2}S(f_2 + g)(f_2 - g) \in \Upsilon(K)$ , then  $g \in \langle f_1 \rangle$ .*
3. *If  $S' = f_1^{-1}f_2^{-1}S(f_1 + g)(f_2 - g) \in \Upsilon(K)$ , then  $S = S'$  and  $g \in \{0, -f_1 + f_2\}$ .*
4. *If  $S' = f_1^{-1}(f_1 + f_2)^{-1}S(f_1 + g)(f_1 + f_2 - g) \in \Upsilon(K)$ , then  $g \in \langle f_2 \rangle$ .*
5. *If  $S' = f_2^{-1}(f_1 + f_2)^{-1}S(f_2 + g)(f_1 + f_2 - g) \in \Upsilon(K)$ , then  $g \in \langle f_1 \rangle$ .*

*Proof.* 1. Since  $f_2^{m-1} \mid S'$  and  $S' \in \Upsilon(K)$ , it follows that  $f_1 + g - (f_1 + f_2) \in \langle f_2 \rangle$ , whence  $g \in \langle f_2 \rangle$ .

2. Analogous to 1.

3. If  $f_1^{m-1} \mid S'$  or  $f_2^{m-1} \mid S'$ , the result follows. Otherwise,  $m \geq 4$  and  $h(S') = m - 1$  imply that  $m = 4$  and  $f_1 + g = f_2 - g = f_1 + f_2$ , a contradiction.

4. Since  $m \geq 3$ , it follows that  $f_1 \mid S'$ . Now we have  $f_2^{m-1} \mid S'$  and  $S' \in \Upsilon(K)$  so that  $(f_1 + f_2 - g) - f_1 \in \langle f_2 \rangle$ , implying  $g \in \langle f_2 \rangle$ , as desired.

5. Analogous to 4. □

**Lemma 3.3.** *Let  $K = C_m \oplus C_m$  with  $m \geq 4$ ,  $g \in K$  and  $S = f_1^{m-1}f_2^{m-1}(f_1 + f_2) \in \Upsilon_{nu}(K)$ .*

1. *If  $S' = f_1^{-2}S(f_1 + g)(f_1 - g) \in \Upsilon_{nu}(K)$ , then  $g = 0$ , and hence  $S = S'$ .*
2. *If  $S' = f_2^{-2}S(f_2 + g)(f_2 - g) \in \Upsilon_{nu}(K)$ , then  $g = 0$ , and hence  $S = S'$ .*
3. *If  $S' = f_1^{-1}f_2^{-1}S(f_1 + g)(f_2 - g) \in \Upsilon_{nu}(K)$ , then  $g \in \{0, -f_1 + f_2\}$ , and hence  $S = S'$ .*
4. *If  $S' = f_1^{-1}(f_1 + f_2)^{-1}S(f_1 + g)(f_1 + f_2 - g) \in \Upsilon_{nu}(K)$ , then  $g \in \{0, f_2\}$ , and hence  $S = S'$ .*
5. *If  $S' = f_2^{-1}(f_1 + f_2)^{-1}S(f_2 + g)(f_1 + f_2 - g) \in \Upsilon_{nu}(K)$ , then  $g \in \{0, f_1\}$ , and hence  $S = S'$ .*

*Proof.* 1. Assume to the contrary that  $g \neq 0$  and  $S \neq S'$ . Since  $S' \in \Upsilon_{nu}(K)$  and  $m \geq 4$ , we get  $f_1 + g = f_1 - g = f_1 + f_2$  and hence  $-2f_2 = 2g = 0$ , a contradiction.

2. - 5. Similar. □

Next we prove two simple structural lemmas which will be our all-purpose tools for turning locally obtained information into global structural conditions on  $S$ . They are also the reason for the hypothesis of  $m$  and  $n$  odd in the Theorem.

**Lemma 3.4.** *Let  $G$  be an abelian group,  $a \in G$  with  $\text{ord}(a) > 2$ , and  $S, T \in \mathcal{F}(G) \setminus \{1\}$  with  $|\text{supp}(S)| \geq |\text{supp}(T)|$ .*

1. *If  $\text{supp}(S) - \text{supp}(T) = \{0\}$ , then  $S = g^{|S|}$  and  $T = g^{|T|}$ , for some  $g \in G$ .*
2. *If  $\text{supp}(S) - \text{supp}(T) \subset \{0, a\}$ , then  $S = g^s(g + a)^{|S|-s}$  and  $T = g^{|T|}$ , for some  $g \in G$  and  $s \in [0, |S|]$ .*
3. *If  $|S|, |T| \geq 2$  and  $\bigcup_{i=1}^2 (\Sigma_i(S) - \Sigma_i(T)) \subset \{0, a\}$ , then either  $S = g^{|S|-1}(g + a)$  and  $T = g^{|T|}$ , or else  $S = g^{|S|}$  and  $T = g^{|T|}$ , for some  $g \in G$ .*

*Proof.* Note that  $\Sigma_1(S) = \text{supp}(S)$  and that all hypotheses imply  $\text{supp}(S) - \text{supp}(T) \subset \{0, a\}$ . Since  $\text{ord}(a) > 2$ , it follows that  $\{0, a\}$  contains no periodic subset, and thus Kneser's Theorem (see e.g., [9, Theorem 5.2.6]) implies that

$$2 \geq |\text{supp}(S) - \text{supp}(T)| \geq |\text{supp}(S)| + |\text{supp}(T)| - 1.$$

Therefore we get  $|\text{supp}(S)| \leq 2$  and  $|\text{supp}(T)| = 1$ . Items 1 and 2 now easily follow. For the proof of part 3, we apply 2, and thus we may assume that  $\text{supp}(S) \subset \{g, (g+a)\}$  and  $T = g^{|T|}$ . Now if item 3 is false, then  $(g+a)^2 \mid S$ , whence

$$2a = ((g+a) + (g+a)) - (g+g) \in \bigcup_{i=1}^2 (\Sigma_i(S) - \Sigma_i(T)) \subset \{0, a\},$$

contradicting that  $\text{ord}(a) > 2$ .  $\square$

**Lemma 3.5.** *Let  $G$  be an abelian group and let  $S \in \mathcal{F}(G)$ .*

1. *If  $k \in [1, |S| - 1]$  and  $|\Sigma_k(S)| \leq 2$ , then  $|\text{supp}(S)| \leq 2$ .*
2. *If  $k \in [2, |S| - 2]$  and  $|\Sigma_k(S)| \leq 2$  and  $\Sigma_k(S)$  is not a coset of a cardinality two subgroup, then either  $S = g^{|S|}$  or  $S = g^{|S|-1}h$ , for some  $g, h \in G$ .*
3. *If  $k \in [1, |S| - 1]$  and  $|\Sigma_k(S)| \leq 1$ , then  $S = g^{|S|}$  for some  $g \in G$ .*

*Proof.* 1. Assume to the contrary that  $|\text{supp}(S)| \geq 3$ , and pick three distinct elements  $x, y, z \in \text{supp}(S)$ . If  $k = |S| - 1$ , then  $\Sigma_{|S|-1}(S) = \sigma(S) - \Sigma_1(S)$  and hence  $|\Sigma_{|S|-1}(S)| = |\text{supp}(S)| \geq 3$ , a contradiction. Therefore  $k \leq |S| - 2$ . Let  $T$  be a subsequence of  $(xyz)^{-1}S$  of length  $|T| = k - 1 \leq |S| - 3$ . Then  $\{x, y, z\} + \sigma(T)$  is a cardinality three subset of  $\Sigma_k(S)$ , a contradiction.

2. By 1, we have  $S = g^{s_1}h^{s_2}$ , with  $s_1, s_2 \in \mathbb{N}_0$ ,  $s_1 \geq s_2$  and  $g, h \in G$  distinct. Assume to the contrary that  $s_2 \geq 2$ . Since  $\Sigma_{|S|-k}(S) = \sigma(S) - \Sigma_k(S)$ , it suffices to consider the case  $k \leq \frac{1}{2}|S|$ , and thus we have  $s_1 \geq \frac{1}{2}|S| \geq k \geq 2$ . Hence the elements  $kg, (k-1)g + h$  and  $(k-2)g + 2h$  are all contained in  $\Sigma_k(S)$ . Thus, since  $|\Sigma_k(S)| \leq 2$  and  $g \neq h$ , it follows  $\text{ord}(h-g) = 2$  and  $\Sigma_k(S) = kg + \{0, h-g\}$ , contradicting that  $\Sigma_k(S)$  is not a coset of a cardinality two subgroup.

3. If the conclusion is false, there are distinct  $x, y \in G$  with  $xy \mid S$ , and then  $\{x, y\} + \sigma(S')$  is a cardinality two subset of  $\Sigma_k(S)$  for any  $S' \mid (xy)^{-1}S$  with  $0 \leq |S'| = k - 1 \leq |S| - 2$ .  $\square$

#### 4. ON THE STRUCTURE OF $\varphi(S)$

**Definition 4.1.** Let  $G = C_{mn} \oplus C_{mn}$  with  $m, n \geq 2$ , let  $S \in \mathcal{A}(G)$  with  $|S| = 2mn - 1$ , and let  $\varphi: G \rightarrow G$  be the multiplication by  $m$  homomorphism. Let

$$\begin{aligned} \Omega'(S) = \Omega' = \{ (W_0, \dots, W_{2m-2}) \in \mathcal{F}(G)^{2m-1} \mid S = W_0 \cdot \dots \cdot W_{2m-2}, \\ \sigma(W_i) \in \text{Ker}(\varphi) \text{ and } |W_i| > 0 \text{ for all } i \in [0, 2m-2] \} \end{aligned}$$

and

$$\Omega(S) = \Omega = \{ (W_0, \dots, W_{2m-2}) \in \Omega' \mid |W_1| = \dots = |W_{2m-2}| = n \}.$$

The elements  $(W_0, \dots, W_{2m-2}) \in \Omega'(S)$  will be called *product decompositions* of  $S$ . If  $W \in \Omega'$ , we implicitly assume that  $W = (W_0, \dots, W_{2m-2})$ .

By Lemma 2.5,  $\Omega \neq \emptyset$ , and if  $W \in \Omega$ , then  $\varphi(W_0), \dots, \varphi(W_{2m-2})$  are minimal zero-sum sequences over  $\varphi(G)$ . Proposition 4.2 below shows that  $\varphi(S)$  is highly structured. We will later in CLAIMS A, B and C of Section 5 (with much effort) show that this structure lifts to the original sequence  $S$ . As this lift will only be ‘near perfect’ (there will be one exceptional term  $x|S$  for which the structure is not shown to lift), we will then, in CLAIM D of Section 5, need Theorem 2.7 to finish the proof of the Theorem.

**Proposition 4.2.** *Let  $G = C_{mn} \oplus C_{mn}$  with  $m, n \geq 2$ , and suppose that  $C_n \oplus C_n$  has Property **B**. Let  $S \in \mathcal{A}(G)$  with  $|S| = 2mn - 1$ , and let  $\varphi: G \rightarrow G$  be the multiplication by  $m$  homomorphism. Then there exist a product decomposition  $(W_0, \dots, W_{2m-2})$  of  $S$  and a basis  $(e_1, e_2)$  of  $\varphi(G)$  such that*

$$(1) \quad \varphi(W_0) = e_1^{n-1} \prod_{\nu=1}^n (x_\nu e_1 + e_2) \quad \text{and} \quad \varphi(W_i) \in \{e_1^n, \prod_{\nu=1}^n (c_{i,\nu} e_1 + e_2)\},$$

where  $x_1, \dots, x_n \in [0, n-1]$ ,  $x_1 + \dots + x_n \equiv 1 \pmod n$ , all  $c_{i,\nu} \in [0, n-1]$ , and  $c_{i,1} + c_{i,2} + \dots + c_{i,n} \equiv 0 \pmod n$  for all  $i \in [1, n]$ . In particular,

$$\varphi(S) = e_1^{\ell n - 1} \prod_{\nu=1}^{2mn - \ell n} (x_\nu e_1 + e_2),$$

where  $\ell \in [1, 2m-1]$  and  $x_\nu \in [0, n-1]$  for all  $\nu \in [1, 2mn - \ell n]$ .

*Proof.* If  $n = 2$ , then it is easy to see (in view of Lemma 2.5) that (1) holds. From now on we assume that  $n \geq 3$ . We distinguish two cases.

CASE 1: For every product decomposition  $W \in \Omega$ , there exist distinct elements  $g_1, g_2 \in \varphi(G)$  such that  $v_{g_1}(\varphi(W_0)) = v_{g_2}(\varphi(W_0)) = n - 1$ .

Let us fix a product decomposition  $W \in \Omega$ . By Lemma 2.3, there is a basis  $(e_1, e'_2)$  of  $\varphi(G)$  such that

$$\varphi(W_0) = e_1^{n-1} \prod_{\nu=1}^n (x_\nu e_1 + e'_2)$$

where  $x_1, \dots, x_n \in [0, n-1]$  and  $x_1 + \dots + x_n \equiv 1 \pmod n$ . Thus, by assumption of CASE 1, it follows that

$$\varphi(W_0) = e_1^{n-1} (x e_1 + e'_2)^{n-1} ((1+x)e_1 + e'_2) \quad \text{with} \quad x \in [0, n-1].$$

As a result,

$$(e_1, e_2) = (e_1, x e_1 + e'_2) = (e_1, e'_2) \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a basis of  $\varphi(G)$  and

$$\varphi(W_0) = e_1^{n-1} e_2^{n-1} (e_1 + e_2).$$

We continue with the following assertion.

**A.** For every  $i \in [1, 2m-2]$ ,  $\varphi(W_i)$  has one of the following forms:

$$e_1^n, e_2^n, (e_1 + e_2)^n, (-e_1 + e_2)^n, (e_1 - e_2)^n, e_1(e_1 + e_2)^{n-2}(e_1 + 2e_2), e_2(e_1 + e_2)^{n-2}(2e_1 + e_2).$$

Suppose that **A** is proved. If the two forms  $(e_1 - e_2)^n$  and  $e_1(e_1 + e_2)^{n-2}(e_1 + 2e_2)$  do not occur, then  $\varphi(W_i)$  has the required form with basis  $(e_1, e_2)$ . If the two forms  $(-e_1 + e_2)^n$  and  $e_2(e_1 + e_2)^{n-2}(2e_1 + e_2)$  do not occur, then  $\varphi(W_i)$  has the required form with basis  $(e_2, e_1)$ . Thus by symmetry, it remains to verify that there are no distinct  $i, j \in [1, 2m-2]$  such that



- (i)  $\varphi(W_i) = e_1(e_1 + e_2)^{n-2}(e_1 + 2e_2)$  and  $\varphi(W_j) = e_2(e_1 + e_2)^{n-2}(2e_1 + e_2)$ ,
- (ii)  $\varphi(W_i) = e_1(e_1 + e_2)^{n-2}(e_1 + 2e_2)$  and  $\varphi(W_j) = (-e_1 + e_2)^n$ , or
- (iii)  $\varphi(W_i) = (e_1 - e_2)^n$  and  $\varphi(W_j) = (-e_1 + e_2)^n$ .

Indeed, if (i) held, then  $(2e_1 + e_2)(e_1 + 2e_2)(e_1 + e_2)^{n-3}$  would be a zero-sum subsequence of  $\varphi(W_i W_j)$  of length  $n - 1$ , contradicting Lemma 2.5. If (ii) held, then  $(-e_1 + e_2)(e_1 + 2e_2)e_2^{n-3}$  would be a zero-sum subsequence of  $\varphi(W_0 W_i W_j)$  of length  $n - 1$ , contradicting Lemma 2.5. Finally, if (iii) held, then  $(e_1 - e_2)(-e_1 + e_2)$  would be a zero-sum subsequence of  $\varphi(W_i W_j)$  of length 2, also contradicting Lemma 2.5. Thus it remains to establish **A** to complete the case. To that end, let  $i \in [1, 2m - 2]$  be arbitrary. Then  $h(\varphi(W_0 W_i)) \geq n - 1$ , and we distinguish three subcases.

CASE 1.1:  $h(\varphi(W_0 W_i)) > n$ .

Then  $v_g(\varphi(W_0 W_i)) > n$  for some  $g \in \{e_1, e_2, e_1 + e_2\}$ . If  $g = e_1 + e_2$ , then  $\varphi(W_i) = (e_1 + e_2)^n$ . Now suppose that  $g \in \{e_1, e_2\}$ , say  $g = e_1$ . Then

$$\varphi(W_0 W_i) = e_2^{n-1}(e_1 + e_2)e_1^n \prod_{\nu=1}^{n-1} (c_\nu e_1 + d_\nu e_2),$$

where  $c_\nu, d_\nu \in [0, n - 1]$  for all  $\nu \in [1, n - 1]$ , and  $c_\nu = 1$  and  $d_\nu = 0$  for some  $\nu \in [1, n - 1]$ . By Lemma 2.5,

$$W'_0 = e_2^{n-1}(e_1 + e_2) \prod_{\nu=1}^{n-1} (c_\nu e_1 + d_\nu e_2)$$

is a minimal zero-sum subsequence of  $\varphi(S)$ . Since  $W'$  contains two distinct elements with multiplicity  $n - 1$  (by assumption of CASE 1), and since  $e_1 | W'_0$ , it follows that either

$$W'_0 = e_1^{n-1}e_2^{n-1}(e_1 + e_2) \quad \text{or} \quad W'_0 = e_1 e_2^{n-1}(e_1 + e_2)^{n-1}.$$

But in the second case, we would get  $\sigma(W'_0) = -2e_2 \neq 0$ . Thus  $W'_0 = e_1^{n-1}e_2^{n-1}(e_1 + e_2)$  and  $\varphi(W_i) = e_1^n$ .

CASE 1.2:  $h(\varphi(W_0 W_i)) = n$ . We distinguish two further subcases.

CASE 1.2.1:  $\varphi(W_i) = g^n$  for some  $g \in \varphi(G) \setminus \{e_1, e_2, e_1 + e_2\}$ .

We set  $g = ce_1 + de_2$  with  $c, d \in [0, n - 1]$ . By Lemmas 2.2 and 2.5, it follows that  $\varphi(W_0)g^{n-1}$  has a zero subsequence  $T$  of length  $|T| = n$  and that  $\varphi(W_i W_0)T^{-1}$  is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n - 1$ , say

$$\varphi(W_i W_0)T^{-1} = e_2^q e_1^r (e_1 + e_2)^s (ce_1 + de_2)^t,$$

where  $q \geq 1, r \geq 1, s \geq 0$  and  $t \in [1, n - 1]$ .

Since  $g \neq e_1 + e_2$ , we infer that  $s \leq 1$ . If  $s = 1$ , then, by assumption of CASE 1, we get

$$2n - 1 = |W_i W_0 T^{-1}| = q + r + s + t \geq 1 + (q + r + t) \geq 1 + (n - 1 + n - 1 + 1) > 2n - 1,$$

a contradiction. Hence  $s = 0$ . Again, by assumption of CASE 1, we have the following possibilities:

- $q = r = n - 1$  and  $t = 1$ .
- $q = t = n - 1$  and  $r = 1$ .
- $q = 1$  and  $r = t = n - 1$ .

If  $q = r = n - 1$  and  $t = 1$ , then  $\sigma(\varphi(W_0W_i)T^{-1}) = 0$  implies that  $g = e_1 + e_2$ , a contradiction. If  $q = t = n - 1$  and  $r = 1$ , then  $\sigma((W_0W_i)T^{-1}) = 0$  implies that  $g = e_1 - e_2$  and  $\varphi(W_i) = (e_1 - e_2)^n$ . Finally, if  $q = 1$  and  $r = t = n - 1$ , then  $\sigma(\varphi(W_0W_i)T^{-1}) = 0$  implies that  $g = -e_1 + e_2$  and  $\varphi(W_i) = (-e_1 + e_2)^n$ .

CASE 1.2.2:  $\mathbf{v}_g(\varphi(W_0W_i)) = n$  for some  $g \in \{e_1, e_2, e_1 + e_2\}$ .

Since  $|W_i| = n$ ,  $\sigma(\varphi(W_i)) = 0$  and  $\mathbf{v}_{e_1+e_2}(\varphi(W_0)) = 1$ , it follows that  $g \neq e_1 + e_2$ . Thus  $g \in \{e_1, e_2\}$ , say  $g = e_1$ . Then

$$\varphi(W_0W_i) = e_2^{n-1}(e_1 + e_2)e_1^n \prod_{\nu=1}^{n-1} (c_\nu e_1 + d_\nu e_2),$$

where  $c_\nu, d_\nu \in [0, n - 1]$  for all  $\nu \in [1, n - 1]$ . By Lemma 2.5 and the assumption of CASE 1.2,

$$W'_0 = e_2^{n-1}(e_1 + e_2) \prod_{\nu=1}^{n-1} (c_\nu e_1 + d_\nu e_2)$$

is a minimal zero-sum subsequence of  $\varphi(S)$  with  $e_1 \nmid W'_0$ . Since  $W'$  contains two distinct elements with multiplicity  $n - 1$  (by the assumption of CASE 1), since  $\sigma(\varphi(W_i)) = 0$ , and since  $e_1 \nmid W'_0$ , it follows that

$$W'_0 = e_2^{n-1}(e_1 + e_2)^{n-1}(e_1 + 2e_2),$$

and thus

$$\varphi(W_i) = e_1(e_1 + e_2)^{n-2}(e_1 + 2e_2).$$

CASE 1.3:  $\mathbf{h}(\varphi(W_0W_i)) = n - 1$ .

Since  $\sigma(\varphi(W_i)) = 0$ , it follows  $\mathbf{v}_g(\varphi(W_0W_i)) \neq n - 1$  for  $g \notin \{e_1, e_2, e_1 + e_2\}$ . Suppose  $\mathbf{v}_{e_1+e_2}(\varphi(W_0W_i)) = n - 1$ . Then

$$\varphi(W_i) = (e_1 + e_2)^{n-2}(c_1e_1 + d_1e_1)(c_2e_1 + d_2e_2),$$

where  $c_1, d_1, c_2, d_2 \in [0, n - 1]$ . By Lemmas 2.2 and 2.5 and the definition of Property **C**,

$$\varphi(W_0W_i)(e_1 + e_2)^{-1}(c_2e_1 + d_2e_2)^{-1}$$

has a zero-sum subsequence  $T$  of length  $|T| = n$  and  $\varphi(W_0W_i)T^{-1}$  is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n - 1$ . Thus it follows, in view of the assumptions of CASE 1 and CASE 1.3, and in view of

$$\varphi(W_0W_i) = e_1^{n-1}e_2^{n-1}(e_1 + e_2)^{n-1}(c_1e_1 + d_1e_2)(c_2e_1 + d_2e_2),$$

that  $\mathbf{h}(T) = n - 1$ , contradicting that  $\sigma(T) = 0$ . So we conclude that

$$(2) \quad \mathbf{v}_g(\varphi(W_0W_i)) < n - 1 \quad \text{for all } g \in \varphi(G) \setminus \{e_1, e_2\}.$$

We set  $\varphi(W_i) = \prod_{\nu=1}^n (c_\nu e_1 + d_\nu e_2)$ , where  $c_\nu, d_\nu \in [0, n - 1]$  for all  $\nu \in [1, n]$ , and pick some  $\lambda \in [1, n]$ . By Lemmas 2.2 and 2.5, it follows that  $\varphi(W_0W_i)(c_\lambda e_1 + d_\lambda e_2)^{-1}$  has a zero-sum subsequence  $T$  of length  $|T| = n$  and that  $\varphi(W_iW_0)T^{-1}$  is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n - 1$ . By assumption of CASE 1 and (2), it follows that

$$\varphi(W_0W_i)T^{-1} = e_1^{n-1}e_2^{n-1}(e_1 + e_2),$$

and thus  $c_\lambda e_1 + d_\lambda e_2 = e_1 + e_2$ . As  $\lambda \in [1, n]$  was arbitrary, this implies that  $\varphi(W_i) = (e_1 + e_2)^n$ , contradicting the hypothesis of CASE 1.3.

CASE 2: There exists a product decomposition  $W \in \Omega$  such that  $\mathbf{v}_g(\varphi(W_0)) = n - 1$  for exactly one element  $g \in \varphi(G)$ .

By Lemma 2.3 and the assumption of CASE 2, there exists a basis  $(e_1, e_2)$  of  $\varphi(G)$  such that

$$\varphi(W_0) = e_1^{n-1} \prod_{\nu=1}^n (x_\nu e_1 + e_2),$$

where  $x_1, \dots, x_n \in [0, n-1]$  and  $x_1 + \dots + x_n \equiv 1 \pmod{n}$  and at most  $n-2$  of the elements  $x_1, \dots, x_n$  are equal. Let  $i \in [1, 2m-2]$  be arbitrary, and let  $\varphi(W_i) = \prod_{\nu=1}^n (c_\nu e_1 + d_\nu e_2)$ , where  $c_\nu, d_\nu \in [0, n-1]$  for all  $\nu \in [1, n]$ . We proceed to show that there exists  $m_i \in \{0, n\}$  such that

$$\varphi(W_i) = e_1^{m_i} \prod_{\nu=1}^{n-m_i} (c_\nu e_1 + e_2),$$

which will complete the proof. We distinguish six subcases.

CASE 2.1:  $\mathbf{h}(\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2)) > n$ .

Then there exists some  $x \in [0, n-1]$  such that (after renumbering if necessary)

$$\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2) = (x e_1 + e_2)^n \prod_{\nu=1}^r (c_\nu e_1 + d_\nu e_2) \prod_{\nu=1}^s (x_\nu e_1 + e_2),$$

where  $r \in [1, n-1]$ ,  $s \in [2, n-1]$  and  $r + s = n$ . Since

$$e_1^{n-1} \prod_{\nu=1}^r (c_\nu e_1 + d_\nu e_2) \prod_{\nu=1}^s (x_\nu e_1 + e_2)$$

is a minimal zero-sum subsequence of  $\varphi(S)$ , Lemma 2.3 implies that  $d_1 = \dots = d_r = 1$ , whence  $\varphi(W_i) = \prod_{\nu=1}^n (c_\nu e_1 + e_2)$ .

CASE 2.2:  $\mathbf{h}(\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2)) = n$ .

If  $(c_1, d_1) = \dots = (c_n, d_n)$  does not hold, then, similar to CASE 2.1, we obtain that  $d_1 = \dots = d_n = 1$ . Therefore  $c_1 = \dots = c_n = c$  and  $d_1 = \dots = d_n = d$  for some  $c, d \in [0, n-1]$ .

Pick some  $\lambda \in [1, n]$ . By Lemmas 2.2 and 2.5, the definition of Property **C**, and the assumption of CASE 2,

$$\varphi(W_0 W_i) (x_\lambda e_1 + e_2)^{-1} (c e_1 + d e_2)^{-1} = (c e_1 + d e_2)^{n-1} e_1^{n-1} \prod_{\nu \in [1, n] \setminus \{\lambda\}} (x_\nu e_1 + e_2)$$

has a zero-sum subsequence  $T$  of length  $n$  and

$$\varphi(W_0 W_i) T^{-1}$$

is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n-1$ . Since  $\varphi(G)$  has Property **B**, we have either

$$e_1^{n-1} \mid \varphi(W_0 W_i) T^{-1} \quad \text{or} \quad (c e_1 + d e_2)^{n-1} \mid \varphi(W_0 W_i) T^{-1}.$$

If  $e_1^{n-1} \mid \varphi(W_0 W_i) T^{-1}$ , then, since  $(x_\lambda e_1 + e_2)(c e_1 + d e_2) \mid \varphi(W_0 W_i) T^{-1}$ , it would follow that  $d = 1$ , whence  $\varphi(W_i) = (c e_1 + e_2)^n$ , as desired. Therefore  $(c e_1 + d e_2)^{n-1} \mid \varphi(W_0 W_i) T^{-1}$ .

Since  $\varphi(W_i)$  is a minimal zero-sum sequence, it follows that

$$n = \text{ord}(c e_1 + d e_2) = \frac{n}{\gcd(c, d, n)},$$

and hence there are  $u, v \in \mathbb{Z}$  such that  $uc + vd \equiv 1 \pmod{n}$ . Thus

$$(e'_1, e'_2) = (ce_1 + de_2, -ve_1 + ue_2) = (e_1, e_2) \cdot \begin{pmatrix} c & -v \\ d & u \end{pmatrix}$$

is a basis of  $\varphi(G)$  and, for some sequence  $Q$  over  $\varphi(G)$ ,

$$\begin{aligned} \varphi(W_0 W_i) T^{-1} &= (ce_1 + de_2)^{n-1} e_1 (x_\lambda e_1 + e_2) Q \\ &= e_1^{n-1} (ue'_1 - de'_2) ((x_\lambda u + v)e'_1 + (c - x_\lambda d)e'_2) Q. \end{aligned}$$

Now Lemma 2.3 implies that  $-d \equiv c - x_\lambda d \pmod{n}$ , whence  $x_\lambda d \equiv c + d \pmod{n}$ . Therefore, since  $\lambda$  was arbitrary, we get

$$d \equiv \sum_{\nu=1}^n x_\nu d \equiv n(c + d) \equiv 0 \pmod{n},$$

and thus  $d = 0$ . If  $c \in [2, n]$ , then  $(ce_1)e_1^{n-c}$  is a zero-sum subsequence of  $\varphi(S)$  of length  $n - c + 1 < n$ , a contradiction. Thus  $c = 1$  and  $\varphi(W_i) = e_1^n$ .

CASE 2.3:  $\mathbf{h}(\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2)) = n - 1$  and  $\mathbf{v}_{e_1}(\varphi(W_i)) \geq 2$ .

After renumbering if necessary, we have

$$\varphi(W_0 W_i) = e_1^{n+1} (xe_1 + e_2)^{n-1} \prod_{\nu=1}^r (x_\nu e_1 + e_2) \prod_{\nu=1}^s (c_\nu e_1 + d_\nu e_2)$$

where  $x \in [0, n-1]$ ,  $r \in [1, n-1]$ ,  $s \in [1, n-2]$  and  $r + s = n - 1$ . By Lemma 2.5,

$$W' = e_1 (xe_1 + e_2)^{n-1} \prod_{\nu=1}^r (x_\nu e_1 + e_2) \prod_{\nu=1}^s (c_\nu e_1 + d_\nu e_2)$$

is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n - 1$ . Since

$$(e_1, e'_2) = (e_1, xe_1 + e_2) = (e_1, e_2) \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a basis of  $\varphi(G)$  and

$$W' = e_1 e_2'^{n-1} \prod_{\nu=1}^r ((x_\nu - x)e_1 + e'_2) \prod_{\nu=1}^s ((c_\nu - x d_\nu)e_1 + d_\nu e'_2),$$

Lemma 2.3 implies that  $x_\nu - x \equiv 1 \pmod{n}$  for all  $\nu \in [1, r]$ . Therefore we get  $(n - r)x + r(x + 1) \equiv \sum_{\nu=1}^n x_\nu \equiv 1 \pmod{n}$ . Hence  $r = 1$  and

$$\varphi(W_0) = e_1^{n-1} (xe_1 + e_2)^{n-1} ((x + 1)e_1 + e_2),$$

a contradiction to our assumption on  $x_1, \dots, x_n$  for CASE 2.

CASE 2.4:  $\mathbf{h}(\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2)) = n - 1$  and  $\mathbf{v}_{e_1}(W_i) = 1$ .

After renumbering if necessary, we get

$$\varphi(W_0 W_i) = e_1^n (xe_1 + e_2)^{n-1} \prod_{\nu=1}^r (x_\nu e_1 + e_2) \prod_{\nu=1}^s (c_\nu e_1 + d_\nu e_2)$$

with  $x \in [0, n-1]$ ,  $r \in [1, n-1]$ ,  $s \in [1, n-1]$  and  $r + s = n$ . By Lemma 2.5,

$$W' = (xe_1 + e_2)^{n-1} \prod_{\nu=1}^r (x_\nu e_1 + e_2) \prod_{\nu=1}^s (c_\nu e_1 + d_\nu e_2)$$

is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n-1$ . Since

$$(e_1, e'_2) = (e_1, xe_1 + e_2) = (e_1, e_2) \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a basis of  $\varphi(G)$  and

$$W' = e_2'^{n-1} \prod_{\nu=1}^r ((x_\nu - x)e_1 + e'_2) \prod_{\nu=1}^s ((c_\nu - xd_\nu)e_1 + d_\nu e'_2),$$

Lemma 2.3 implies that

$$(3) \quad x_1 - x \equiv \dots \equiv x_r - x \equiv c_1 - xd_1 \equiv \dots \equiv c_s - xd_s \pmod{n}.$$

If  $d_1 = \dots = d_s = 1$ , then  $\varphi(W_i) = \prod_{\nu=1}^n (c_\nu e_1 + e_2)$ , as desired. Therefore there is some  $\nu \in [1, s]$  with  $d_\nu \neq 1$ , say  $\nu = s$ . Hence, since  $\sigma(W_i) = 0$ , it follows that there is also another  $\nu' \in [1, s]$  with  $d_{\nu'} \neq 1$  and  $s = \nu \neq \nu'$ . Thus, by Lemmas 2.2 and 2.5 and the definition of Property **C**,

$$\varphi(W_0 W_i) e_1^{-1} (c_s e_1 + d_s e_2)^{-1}$$

has a zero-sum subsequence  $T$  of length  $|T| = n$  and  $\varphi(W_0 W_i) T^{-1}$  is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n-1$ . Since  $\varphi(G)$  has Property **B**, it follows that either

$$e_1^{n-1} \mid \varphi(W_0 W_i) T^{-1} \quad \text{or} \quad (xe_1 + e_2)^{n-1} \mid \varphi(W_0 W_i) T^{-1}.$$

If  $e_1^{n-1} \mid \varphi(W_0 W_i) T^{-1}$ , then, since  $(c_s e_1 + d_s e_2) \mid \varphi(W_0 W_i) T^{-1}$  and  $(x_j e_1 + e_2) \mid \varphi(W_0 W_i) T^{-1}$  for some  $j \in [1, n]$ , Lemma 2.3 implies that  $d_s = 1$ , a contradiction. Therefore  $(xe_1 + e_2)^{n-1} \mid \varphi(W_0 W_i) T^{-1}$ . Thus, for some sequence  $Q$  over  $\varphi(G)$ , we have

$$\varphi(W_0 W_i) T^{-1} = (xe_1 + e_2)^{n-1} e_1 (c_s e_1 + d_s e_2) Q.$$

Since

$$(e_1, e'_2) = (e_1, xe_1 + e_2) = (e_1, e_2) \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a basis of  $\varphi(G)$  and

$$\varphi(W_0 W_i) T^{-1} = e_1 e_2'^{n-1} ((c_s - xd_s) e_1 + d_s e'_2) Q,$$

Lemma 2.3 implies that  $c_s - xd_s = 1$ . Thus it follows from (3) that  $x_1 \equiv \dots \equiv x_r \equiv x+1 \pmod{n}$ . Therefore we get  $(n-r)x + r(x+1) \equiv \sum_{\nu=1}^n x_\nu \equiv 1 \pmod{n}$ . Hence  $r = 1$  and

$$\varphi(W_0) = e_1^{n-1} (xe_1 + e_2)^{n-1} ((x+1)e_1 + e_2),$$

a contradiction to our assumption on  $x_1, \dots, x_n$  for CASE 2.

CASE 2.5:  $\mathbf{h}(\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2)) = n-1$  and  $\mathbf{v}_{e_1}(\varphi(W_i)) = 0$ .

If  $d_1 = \dots = d_n = 1$ , then the assertion follows. Therefore there is some  $\nu \in [1, n]$  with  $d_\nu \neq 1$ , say  $\nu = n$ . Since  $d_1 + \dots + d_n \equiv 0 \pmod{n}$ , we may also assume that  $d_{n-1} \neq 1$ . We distinguish two subcases.

CASE 2.5.1:  $\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2)$  contains two distinct elements with multiplicity  $n-1$ , say  $xe_1 + e_2$  and  $ye_1 + e_2$ , where  $x, y \in [0, n-1]$ .

Then

$$\varphi(W_i) = (xe_1 + e_2)^r (ye_1 + e_2)^s (c_{n-1}e_1 + d_{n-1}e_2)(c_n e_1 + d_n e_2)$$

and

$$\prod_{\nu=1}^n (x_\nu e_1 + e_2) = (xe_1 + e_2)^{n-1-r} (ye_1 + e_2)^{n-1-s},$$

where  $r, s \in [1, n-3]$  and  $r+s = n-2 \geq 2$ . By Lemmas 2.2 and 2.5,  $\varphi(W_0 W_i)(c_n e_1 + d_n e_2)^{-1}$  has a zero-sum subsequence  $T$  of length  $|T| = n$  and  $\varphi(W_0 W_i)T^{-1}$  is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n-1$ . Since  $\varphi(G)$  has Property **B**, it follows that

$$\nu_g(\varphi(W_i W_0)T^{-1}) = n-1 \quad \text{for some } g \in \{e_1, xe_1 + e_2, ye_1 + e_2\}.$$

Clearly, we have

$$e_1(xe_1 + e_2)(ye_1 + e_2)(c_n e_1 + d_n e_2) \mid \varphi(W_0 W_i)T^{-1}.$$

Since  $d_n \neq 1$ , Lemma 2.3 implies that  $g \neq e_1$ . Thus w.l.o.g.  $g = xe_1 + e_2$ . Consequently, for some sequence  $Q$  over  $\varphi(G)$ , we have

$$\varphi(W_0 W_i)T^{-1} = (xe_1 + e_2)^{n-1} e_1 (ye_1 + e_2) Q.$$

As before,

$$(e_1, e'_2) = (e_1, xe_1 + e_2) = (e_1, e_2) \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a basis of  $\varphi(G)$  and

$$\varphi(W_0 W_i)T^{-1} = e'_2{}^{n-1} e_1 ((y-x)e_1 + e'_2) Q.$$

Now we obtain a contradiction as in CASE 2.3.

CASE 2.5.2:  $\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2)$  contains exactly one element with multiplicity  $n-1$ , say  $xe_1 + e_2$  where  $x \in [0, n-1]$ .

After renumbering if necessary, we get

$$\varphi(W_0 W_i) = e_1^{n-1} (xe_1 + e_2)^{n-1} \prod_{\nu=1}^r (c_\nu e_1 + d_\nu e_2) \prod_{\nu=1}^s (x_\nu e_1 + e_2),$$

where  $r \in [1, n-1]$ ,  $s \in [2, n-1]$  and  $r+s = n+1$ . If  $d_1 = \dots = d_r = 1$ , then the assertion follows. So after renumbering again, we suppose that  $d_r \neq 1$ . Let  $\lambda \in [1, s]$ .

By Lemmas 2.2 and 2.5, the definition of Property **C**, and the assumption of CASE 2.5.2,

$$\varphi(W_0 W_i)(c_r e_1 + d_r e_2)^{-1} (x_\lambda e_1 + e_2)^{-1}$$

has a zero-sum subsequence  $T$  of length  $|T| = n$  and  $\varphi(W_0 W_i)T^{-1}$  is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n-1$ . Since  $\varphi(G)$  has Property **B**, it follows that

$$\nu_g(\varphi(W_0 W_i)T^{-1}) = n-1 \quad \text{for some } g \in \{e_1, xe_1 + e_2, \dots\}.$$

Clearly, we have

$$e_1(xe_1 + e_2)(c_r e_1 + d_r e_2)(x_\lambda e_1 + e_2) \mid \varphi(W_0 W_i)T^{-1}.$$

Since  $d_r \neq 1$ , Lemma 2.3 implies that  $g \neq e_1$ , and hence  $g = xe_1 + e_2$ . Thus, for some sequence  $Q$  over  $\varphi(G)$ , we have

$$\varphi(W_0W_i)T^{-1} = (xe_1 + e_2)^{n-1}e_1(x_\lambda e_1 + e_2)Q.$$

As before,

$$(e_1, e'_2) = (e_1, xe_1 + e_2) = (e_1, e_2) \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a basis of  $\varphi(G)$  and

$$\varphi(W_0W_i)T^{-1} = e_2'^{n-1}e_1((x_\lambda - x)e_1 + e_2')Q.$$

Hence Lemma 2.3 implies that  $1 \equiv x_\lambda - x \pmod{n}$ . As  $\lambda \in [1, s]$  was arbitrary, it follows that  $x_1 \equiv \dots \equiv x_s \equiv x + 1 \pmod{n}$ , and, as in CASE 2.3, we obtain a contradiction.

CASE 2.6:  $\mathbf{h}(\varphi(W_i) \prod_{\nu=1}^n (x_\nu e_1 + e_2)) < n - 1$ .

Let  $\lambda \in [1, n]$  be arbitrary. By Lemmas 2.2 and 2.5,

$$\varphi(W_0W_i)(c_\lambda e_1 + d_\lambda e_2)^{-1}$$

has a zero-sum subsequence  $T$  of length  $|T| = n$ , and

$$\varphi(W_0W_i)T^{-1}$$

is a minimal zero-sum subsequence of  $\varphi(S)$  of length  $2n - 1$ . Since  $\varphi(G)$  has Property **B**, it follows that  $e_1^{n-1}$  divides  $\varphi(W_0W_i)T^{-1}$ . Furthermore there is some  $\nu \in [1, n]$  such that

$$(x_\nu e_1 + e_2)(c_\lambda e_1 + d_\lambda e_2) | \varphi(W_0W_i)T^{-1}.$$

Thus Lemma 2.5 implies that either  $d_\lambda = 1$  or  $(c_\lambda, d_\lambda) = (1, 0)$ . Thus, since  $\lambda \in [1, n]$  was arbitrary and  $\sigma(\varphi(W_i)) = 0$ , we must either have  $d_\lambda = 1$  for all  $\lambda \in [1, n]$  or  $(c_\lambda, d_\lambda) = (1, 0)$  for all  $\lambda \in [1, n]$ , and so either  $\varphi(W_i) = e_1^n$  or  $\varphi(W_i) = \prod_{\nu=1}^n (c_\nu e_1 + e_2)$ , as desired  $\square$

## 5. PROOF OF THE THEOREM

Let  $G = C_{mn} \oplus C_{mn}$ , with  $m, n \geq 3$  odd,  $mn > 9$  and w.l.o.g.  $m \geq 5$ , such that Property **B** holds both for  $C_m \oplus C_m$  and  $C_n \oplus C_n$ . Let  $S \in \mathcal{A}(G)$  be a minimal zero-sum sequence of length  $|S| = 2mn - 1$ . The sequence  $S$  will remain fixed throughout the rest of this section. Our goal is to show that  $S$  contains an element with multiplicity  $mn - 1$  (in other words,  $\mathbf{h}(S) = mn - 1$ ). We proceed in the following way:

- First, using Proposition 4.2, we establish the setting and some detailed notation necessary to formulate the key ideas of the proof.
- Next, we proceed with four lemmas, Lemmas 5.1, 5.2, 5.3 and 5.4, that collect several arguments used repeatedly in the proof.
- Then we divide the main part of the proof into four claims, CLAIMS A, B, C and D, where in CLAIM D we finally show that  $\mathbf{h}(S) = mn - 1$ .

### The Setting and Key Definitions

Since  $S$  is fixed, we write  $\Omega'$  and  $\Omega$  instead of  $\Omega'(S)$  and  $\Omega(S)$  (see Definition 4.1). Recall that Lemma 2.3.3 implies that  $\text{ord}(x) = mn$  for all  $x \in \text{supp}(S)$ . Let  $\varphi: G \rightarrow G$  denote the multiplication by  $m$  map. Then  $\text{Ker}(\varphi) = nG \cong C_m \oplus C_m$  and  $\varphi(G) = mG \cong C_n \oplus C_n$ .

Let  $\Omega_0 \subset \Omega$  be all those  $W \in \Omega$  for which there exists a basis  $(me_1, me_2)$  of  $\varphi(G)$ , where  $e_1, e_2 \in G$ , such that  $\varphi(W_0) = (me_1)^{n-1} \prod_{\nu=1}^n (x_\nu me_1 + me_2)$ , where  $x_1, \dots, x_n \in \mathbb{Z}$  with  $x_1 + \dots + x_n \equiv 1 \pmod n$ , and such that for every  $i \in [1, 2m-2]$ ,  $\varphi(W_i)$  is either of the form  $\varphi(W_i) = (me_1)^n$ , or of the form  $\varphi(W_i) = \prod_{\nu=1}^n (y_{i,\nu} me_1 + me_2)$ , where  $y_{i,1}, \dots, y_{i,n} \in \mathbb{Z}$  with  $y_{i,1} + \dots + y_{i,n} \equiv 0 \pmod n$ . By Proposition 4.2,  $\Omega_0$  is nonempty.

Let  $W \in \Omega'$ , and define  $\tilde{\sigma}(W) = \prod_{\nu=0}^{2m-2} \sigma(W_\nu) \in \mathcal{F}(\text{Ker}(\varphi))$ . Since  $S \in \mathcal{A}(G)$ , it follows that  $\tilde{\sigma}(W) \in \mathcal{A}(\text{Ker}(\varphi))$ . Thus, since Property **B** holds for  $\text{Ker}(\varphi)$ , it follows that  $\tilde{\sigma}(W) \in \Upsilon(\text{Ker}(\varphi))$ . Partition  $\Omega_0 = \Omega_0^u \cup \Omega_0^{nu}$  by letting  $\Omega_0^u$  be those  $W \in \Omega_0$  with  $\tilde{\sigma}(W) \in \Upsilon_u(\text{Ker}(\varphi))$ , and letting  $\Omega_0^{nu}$  be those  $W \in \Omega_0$  with  $\tilde{\sigma}(W) \in \Upsilon_{nu}(\text{Ker}(\varphi))$ .

Let  $W \in \Omega_0$ , let  $(me_1, me_2)$  be a basis of  $\varphi(G)$  satisfying the definition of  $\Omega_0$ , with  $e_1, e_2 \in G$ , and let  $(f_1, f_2)$  be a basis for  $\text{Ker}(\varphi)$  such that  $\tilde{\sigma}(W)$  can be written as in the definition of  $\Upsilon(\text{Ker}(\varphi))$ . Let  $S_1$  be the subsequence of  $S$  consisting of all terms  $x$  with  $\varphi(x) = me_1$ , and define  $S_2$  by  $S = S_1 S_2$ . Let  $I \subset \mathbb{Z}$  be an interval of length  $n$ . Then each term  $x$  of  $S_1$  has a unique representation of the form  $x = e_1 + ng$ , with  $ng \in \text{Ker}(\varphi)$  (where  $g \in G$ ), and each term  $x$  of  $S_2$  has a unique representation of the form  $x = ae_1 + e_2 + ng$ , with  $a \in I$  and  $ng \in \text{Ker}(\varphi)$  (where  $g \in G$ ). Define  $\psi(x) = ng \in \text{Ker}(\varphi)$  and, for  $x \in \text{supp}(S_2)$ , define  $\iota(x) = a \in I \subset \mathbb{Z}$ . We set  $\psi(x) = \psi_1(x) + \psi_2(x)$ , where  $\psi_1(x) \in \langle f_1 \rangle$  and  $\psi_2(x) \in \langle f_2 \rangle$ . If  $y \in \text{Ker}(\varphi)$ , with  $y = y_1 f_1 + y_2 f_2$ , then we also use  $\psi_i(y)$  to denote  $y_i f_i$ . Note that, for  $x \in \text{supp}(S_1)$ , the value of  $\psi(x)$  depends upon the choice of  $(e_1, e_2)$ , and that, for  $x \in \text{supp}(S_2)$ , the values of  $\psi(x)$  and  $\iota(x)$  depend upon the choice of  $(e_1, e_2)$  and  $I$ . We will frequently need to vary the underlying choices for  $(e_1, e_2)$  and  $I$ , and each time we do so the corresponding values of  $\psi$  and  $\iota$  will be affected. All maps will be extended to sequences as explained before Definition 2.1.

Let  $\mathcal{A}_1(W)$  be those  $W_i$  either with  $i = 0$  or  $\varphi(W_i) = (me_1)^n$ , let  $\mathcal{A}_2(W)$  be all remaining  $W_i$  as well as  $W_0$ , and let  $\mathcal{A}_i^*(W) = \mathcal{A}_i(W) \setminus \{W_0\}$  for  $i \in \{1, 2\}$ . If  $W \in \Omega_0^u$ , let  $\mathcal{C}_0(W)$  be all those  $W_i$  with  $\mathbf{v}_{\sigma(W_i)}(\tilde{\sigma}(W)) < m-1$ , let  $\mathcal{C}_1(W)$  be all remaining  $W_i$ , and let  $\mathcal{C}_i^*(W) = \mathcal{C}_i(W) \setminus \{W_0\}$  for  $i \in \{0, 1\}$ . If  $W \in \Omega_0^{nu}$ , let  $\mathcal{C}_0(W)$  be the unique  $W_i$  with  $\mathbf{v}_{\sigma(W_i)}(\tilde{\sigma}(W)) < m-1$ , and divide the remaining  $2m-2$  blocks  $W_i$  into either  $\mathcal{C}_1(W)$  or  $\mathcal{C}_2(W)$  depending on the value of  $\sigma(W_i)$ ; analogously define  $\mathcal{C}_i^*(W)$  for  $i \in \{0, 1, 2\}$ . When the context is clear, the  $W$  will be omitted from the notation. We regard the elements  $W_i, W_j \in \mathcal{A}_1$  as distinct when  $i \neq j$ , follow the same convention for all other similar collections of  $W_i$ , and will refer to them as blocks.

We further subdivide  $W_0 = W_0^{(1)} W_0^{(2)}$  with  $W_0^{(1)} = \gcd(W_0, S_1)$  and  $W_0^{(2)} = \gcd(W_0, S_2)$ , and for a pair of subsequences  $X$  and  $Y$  with  $XY \mid S_2$ , we define  $\epsilon'(X, Y)$  to be the integer in  $[1, n]$  congruent to  $\sigma(\iota(X)) - \sigma(\iota(Y))$  modulo  $n$ , and define  $\epsilon(X, Y)$  to be the integer such that

$$n - \epsilon'(X, Y) + \sigma(\iota(X)) - \sigma(\iota(Y)) = \epsilon(X, Y)n.$$

The main idea of the proof is to swap individual terms contained in the blocks of  $W \in \Omega_0$  in such a way so as to maintain that the resulting product decomposition still lies in  $\Omega'$ . Using the lemmas from Section 3, we will then derive information about the possible values of  $\psi$  and  $\iota$  obtained on the terms



that have been swapped. The next three paragraphs detail the three major types of swaps that we will use.

If  $U, V \in \mathcal{A}_1$  are distinct (thus  $U = W_i$  and  $V = W_j$  for some  $i$  and  $j$  distinct), then we may exchange any subsequence  $X|U$  for a subsequence  $Y|V$  with  $|Y| = |X|$  (if  $U = W_0$ , then  $X$  must additionally lie within  $W_0^{(1)}$ , and likewise for  $V$ ) and the resulting product decomposition  $W'$  will still lie in  $\Omega_0$ , equal to  $W$  except that the blocks  $U$  and  $V$  of  $W$  have been replaced by the blocks  $U' := X^{-1}UY$  and  $V' := Y^{-1}VX$ . Moreover,

$$(4) \quad \sigma(V') = \sigma(V) + \sigma(\psi(X)) - \sigma(\psi(Y)).$$

We refer to this as a *type I swap*.

If  $V \in \mathcal{A}_2^*$ , and  $Y|V$  and  $X|W_0^{(2)}$  are subsequences with  $|X| = |Y|$ , then by exchanging the sequence  $Y|V$  for the sequence  $RX|W_0$ , where  $R|W_0^{(1)}$  is any subsequence with  $|R| = n - \epsilon'(X, Y)$ , we obtain a product decomposition  $W'$  that still lies in  $\Omega'$ , equal to  $W$  except that the blocks  $V$  and  $W_0$  of  $W$  have been replaced by the blocks  $V' := Y^{-1}VXR$  and  $W'_0 := R^{-1}X^{-1}W_0Y$ . Moreover,

$$(5) \quad \sigma(V') = \sigma(V) + \epsilon(X, Y)ne_1 + \sigma(\psi(X)) - \sigma(\psi(Y)) + \sigma(\psi(R)).$$

We refer to this as a *type II swap*.

If  $U, V \in \mathcal{A}_2$  are distinct, then we may exchange any subsequence  $X|U$  for a subsequence  $Y|V$  with  $|Y| = |X|$  and  $\sigma(\iota(X)) = \sigma(\iota(Y))$  (and if  $U = W_0$ , then  $X$  must additionally lie within  $W_0^{(2)}$ , and likewise for  $V$ ) and the resulting product decomposition  $W'$  will still lie in  $\Omega_0$ , equal to  $W$  except that the blocks  $U$  and  $V$  of  $W$  have been replaced by the blocks  $U' := X^{-1}UY$  and  $V' := Y^{-1}VX$ . Moreover,

$$(6) \quad \sigma(V') = \sigma(V) + \sigma(\psi(X)) - \sigma(\psi(Y)).$$

We refer to this as a *type III swap*.

We will often also have need to change from  $W \in \Omega_0$  to another  $W' \in \Omega_0$ . One common way that this will be done will be to find  $U \in \mathcal{A}_2^*$  and  $X|UW_0^{(2)}$  ( $X$  will often be a single element dividing  $U$ ). Then  $|X^{-1}UW_0^{(2)}| = 2n - |X|$ . If there is an  $n$ -term subsequence  $U'|X^{-1}UW_0^{(2)}$  with  $\sigma(U') \in \text{Ker}(\varphi)$  (as is guaranteed by Theorem 2.6.1 in case  $|X| = 1$ ), then, defining  $W'_0$  by  $W'_0U' = W_0U$ , we obtain a new product decomposition  $W' \in \Omega_0$  by replacing the blocks  $W_0$  and  $U$  by  $W'_0$  and  $U'$ . Moreover,  $X|W'_0^{(2)}$ . We refer to such a procedure as *pulling  $X$  up into the new product decomposition  $W'$* .

All of the above procedures result in a new product decomposition  $W' \in \Omega'$ , and we will always assume  $W' = (W'_0, \dots, W'_{2m-2})$ , with  $W'_k = W_k$  for all blocks  $W_k$  not involved in the procedure, and with  $W'_i$  and  $W'_j$  defined as above for the two blocks  $W_i$  and  $W_j$  involved in the procedure.

### Four Lemmas

We will often only consider  $W \in \Omega_0^{nu}$  when  $\Omega_0^u = \emptyset$  (with one exception in CASE 3 of CLAIM C). The reason for this is to ensure that, if a swapping procedure applied to  $W$  results in a new product decomposition  $W' \in \Omega_0$ , then  $W' \in \Omega_0^{nu}$  is guaranteed, and hence the more powerful Lemma 3.3 is available (instead of the weaker Lemma 3.2).

The following lemma will be used in CASE 3 of CLAIM C to avoid having to consider a  $W'' \in \Omega_0^{nu}$  when  $\Omega_0^u \neq \emptyset$ .

**Lemma 5.1.** *Let  $W \in \Omega_0^u$ ,  $U \in \mathcal{C}_1$  and  $V_1, V_2 \in \mathcal{C}_0$  be distinct. Suppose there exist  $X|U$  and  $Y_1|V_1$  such that swapping  $X$  for  $Y_1$  yields a new product decomposition  $W' \in \Omega'$  with the new block  $U' = X^{-1}UY_1$  in  $W'$  having  $\sigma(U') \neq \sigma(U)$ . If  $Y_2|Y_1^{-1}V_1$  and  $Z|V_2$  are nontrivial subsequences such that swapping  $Y_2$  for  $Z$  in  $W$  yields a new product decomposition  $W'' \in \Omega_0$ , then  $W'' \in \Omega_0^u$ .*

*Proof.* Assume by contradiction that  $W'' \in \Omega_0^{nu}$ , so that w.l.o.g.  $\tilde{\sigma}(W'') = f_1^{m-1}f_2^{m-1}(f_1 + f_2)$  with  $\sigma(U) = f_1$  (since  $\sigma(U)$  is a maximal multiplicity term in  $\tilde{\sigma}(W)$  and all blocks involved in the swap resulting in  $W''$  are not of maximal multiplicity, it follows that  $\sigma(U'') = \sigma(U)$  must be a maximal multiplicity term in  $\tilde{\sigma}(W'')$  as well). Since  $m \geq 4$  (so that  $f_2f_2^{m-1}|\tilde{\sigma}(W)$ ), let  $\sigma(V_1) = Cf_1 + f_2$  with  $C \in [0, m-1]$ . By hypothesis, we may swap  $Y_1|V_1'' = Y_2^{-1}V_1Z$  for  $X|U'' = U$  to obtain a new product decomposition  $W''' \in \Omega'$ , with new respective terms  $V_1'''$  and  $U'''$ . Since (by hypothesis) swapping  $X$  for  $Y_1$  in  $W$  yields a new product decomposition  $W' \in \Omega'$  such that the new block  $U' = X^{-1}UY_1$  in  $W'$  has  $\sigma(U') \neq \sigma(U)$ , it follows from Lemma 3.1.2 that  $\sigma(U''') = \sigma(U') = Cf_1 + f_2$  and  $\sigma(V_1''') = \sigma(V_1'') + (1-C)f_1 - f_2$ .

Suppose  $\sigma(V_1'') = f_2$ . Then, from the above paragraph, we conclude that

$$\tilde{\sigma}(W''') = f_2^{m-2}(f_1 + f_2)((1-C)f_1)f_1^{m-2}(Cf_1 + f_2).$$

Thus, since  $\tilde{\sigma}(W''') \in \Upsilon(\text{Ker}(\varphi))$  and  $m \geq 4$ , it follows that  $C = 0$ , whence  $\sigma(V_1'') = f_2 = Cf_1 + f_2 = \sigma(V_1)$ . However, this implies that  $\tilde{\sigma}(W) = \tilde{\sigma}(W'') \in \Upsilon_0^{nu}$ , contrary to  $W \in \Omega_0^u$ . So we may assume instead that  $\sigma(V_1'') = f_1 + f_2$  (note  $\sigma(V_1'') \neq f_1$ , since  $\sigma(U) = f_1$ ,  $U \in \mathcal{C}_1(W)$  and no terms from  $\mathcal{C}_1(W)$  were involved in the swap resulting in  $W''$ ).

In this case, we instead conclude that

$$\tilde{\sigma}(W''') = f_2^{m-1}((2-C)f_1)f_1^{m-2}(Cf_1 + f_2).$$

Thus, since  $\tilde{\sigma}(W''') \in \Upsilon(\text{Ker}(\varphi))$  and  $m \geq 3$ , we conclude that  $C = 1 = 2 - C$ , and once more  $\sigma(V_1'') = \sigma(V_1)$ , yielding the same contradiction as in the previous paragraph, completing the lemma.  $\square$

The next two lemmas will often be used in conjunction, and will form one of our main swapping strategy arguments used for CLAIMS A and B. Note that Lemma 5.2(i) gives a strong structural description as well as a term of multiplicity at least  $(|\mathcal{D}_1| + 1)n - 1$  in  $S$ , while Lemma 5.2(ii) allows us to invoke Lemma 5.3.

**Lemma 5.2.** *Let  $W \in \Omega_0$  and, if  $\Omega_0^u \neq \emptyset$ , assume that  $W \in \Omega_0^u$ . Let  $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{A}_2^*$  be such that, for each (relevant)  $i \in [0, 2]$ , there do not exist  $U \in \mathcal{D}_1$  and  $V \in \mathcal{D}_2$  with  $U, V \in \mathcal{C}_i$ . If either*

- (a)  $|\mathcal{D}_1| \geq 1$  and every type III swap between  $x|W_0^{(2)}$  and  $y|W_j$ , with  $W_j \in \mathcal{D}_1$  and  $\iota(x) = \iota(y)$ , results in a new product decomposition  $W'$  with  $\sigma(W'_0) = \sigma(W_0)$ , or
- (b)  $|\mathcal{D}_1| \geq 2$  and  $|\mathcal{D}_2| \geq 1$ ,

then one of the following two statements hold:

- (i) There exist  $x_0|W_0^{(2)}$ ,  $g \in I$  and  $\alpha \in \text{Ker}(\varphi)$  such that  $\iota(x_0) \equiv g+1 \pmod n$ ,  $\iota(x) = g$  and  $\psi(x) = \alpha$ , for all  $x|x_0^{-1}W_0^{(2)} \prod_{V \in \mathcal{D}_1} V$ .
- (ii) There exist  $W_j \in \mathcal{D}_1$ ,  $X|W_0^{(2)}$  and  $Y|W_j$  such that  $|X| = |Y|$  and  $\epsilon'(X, Y) \notin \{1, n\}$ .

*Proof.* We assume that (ii) fails and show that (i) holds. If  $W_0 \in \mathcal{C}_0$ , then choose  $f_2$  such that  $\sigma(W_0) = f_1 + f_2$ ; if  $W \in \Omega_0^{nu}$ , then choose  $f_2$  such that  $\tilde{\sigma}(W) = f_1^{m-1}f_2^{m-1}(f_1 + f_2)$  (note, in case  $W_0 \in \mathcal{C}_0$  and

$W \in \Omega_0^{nu}$ , that this choice of  $f_2$  agrees with the previous choice), and assume  $\mathcal{C}_1$  consists of those  $W_i$  with  $\sigma(W_i) = f_1$ ; and if  $W_0 \notin \mathcal{C}_0$ , then w.l.o.g. assume  $W_0 \in \mathcal{C}_1$ .

Applying Lemma 3.4.3 to  $\iota(W_0^{(2)})$  and each  $\iota(V)$  with  $V \in \mathcal{D}_1$ , with both sequences considered modulo  $n$  (since (ii) fails, the hypothesis of Lemma 3.4.3 holds with  $\{0, a\}$  equal to  $\{n, 1\}$  modulo  $n$ ), we conclude, in view of  $\sigma(\iota(W_0^{(2)})) \equiv 1 \pmod n$  (and hence  $|\text{supp}(\iota(W_0^{(2)}))| > 1$ ), that there exist  $x_0|W_0^{(2)}$  and  $g \in I$  such that  $\iota(x_0) \equiv g + 1 \pmod n$  and  $\iota(x) = g$  for all  $x|x_0^{-1}W_0^{(2)} \prod_{V \in \mathcal{D}_1} V$ . If (a) holds, then performing type III swaps between  $W_0$  and the  $V \in \mathcal{D}_1$  completes the proof. Therefore assume (a) fails and (b) holds instead.

CASE 1:  $W_0 \in \mathcal{C}_0$ .

Thus, since  $|\mathcal{D}_1|, |\mathcal{D}_2| \geq 1$ , let  $U \in \mathcal{A}_2^* \cap (\mathcal{D}_1 \cup \mathcal{D}_2)$  with  $\sigma(U) = f_1$  and let  $V \in \mathcal{A}_2^* \cap (\mathcal{D}_1 \cup \mathcal{D}_2)$  with  $\sigma(V) = Cf_1 + f_2$  for some  $C \in \mathbb{Z}$ . Performing a type II swap between some fixed  $u|U$  and each  $x|x_0^{-1}W_0^{(2)}$  (using the same fixed subsequence  $R|W_0^{(1)}$  in every swap, which is possible since  $\iota(x) = g$  for all  $x|x_0^{-1}W_0^{(2)}$ ), we conclude from either Lemma 3.1.2 (since  $\sigma(W_0) = f_1 + f_2$ ) or Lemma 3.2.4 that  $\psi_1$  is constant on  $x_0^{-1}W_0^{(2)}$ . Likewise performing a type II swap between some fixed  $v|V$  and each  $x|x_0^{-1}W_0^{(2)}$ , we conclude from either Lemma 3.1.3 or Lemma 3.2.5 that  $\psi_2$  is constant on  $x_0^{-1}W_0^{(2)}$ . Consequently,  $\psi(x) = \alpha$  (say) for all  $x|x_0^{-1}W_0^{(2)}$ .

Suppose  $W \in \Omega_0^{nu}$ . Then  $\mathcal{D}_1 \subset \mathcal{A}_2^* \cap \mathcal{C}_i$ , for some  $i \in \{1, 2\}$  (in view of the hypotheses of CASE 1 and the lemma), and performing type III swaps between the  $Z \in \mathcal{D}_1$ , we conclude, in view of  $|\mathcal{D}_1| \geq 2$  and Lemma 3.3.1 or 3.3.2, that  $\psi(x) = \alpha'$  (say) for all  $x|\prod_{V \in \mathcal{D}_1} V$ . Further applying type III swaps between  $W_0$  and any  $Z \in \mathcal{D}_1$ , we conclude from Lemma 3.4.3 and either Lemma 3.3.4 or 3.3.5 that  $\alpha = \alpha'$ , completing the proof. So we may assume  $W \in \Omega_0^u$ .

If  $\mathcal{D}_1 \subset \mathcal{C}_1$ , then repeating the argument of the previous paragraph using Lemma 3.1 in place of Lemma 3.3 completes the proof. Therefore we may assume  $\mathcal{D}_1 \subset \mathcal{C}_0$ . Let  $Z \in \mathcal{D}_1$  and  $z|Z$ . We proceed to show  $\psi(z) = \alpha$ , which, since  $z|Z \in \mathcal{D}_1$  is arbitrary, will complete the proof.

If performing a type III swap between  $z|Z$  and some  $x|x_0^{-1}W_0^{(2)}$  results in a new product decomposition  $W' \in \Omega_0^u$ , then  $W'_0 \in \mathcal{C}_0$  (as both  $W_0, Z \in \mathcal{C}_0$ ) and, repeating the arguments of the first paragraph of CASE 1 this time for  $W'$ , we conclude that  $\psi(z) = \alpha$ . If  $W' \in \Omega_0^{nu}$ , then we can choose a new  $f_2$  such that  $\tilde{\sigma}(W') = f_1^{m-1}f_2^{m-1}(f_1 + f_2)$ . If also  $W'_0 \in \mathcal{C}_0$ , then  $\sigma(W'_0) = f_1 + f_2$ , and repeating the arguments of the first paragraph for  $W'$  shows  $\psi(z) = \alpha$ . Therefore suppose  $W' \in \Omega_0^{nu}$  and  $\sigma(W'_0) = f_2$ . In view of Lemma 3.1.3, we have  $\alpha - \psi(z) \in \langle f_1 \rangle$ . However, if  $\alpha \neq \psi(z)$ , then performing a type II swap between some  $y|U' = U$  and both  $z|W'_0$  and  $z'|W'_0$ , where  $\iota(z') = g$  and  $\psi(z') = \alpha$ , we conclude from Lemma 3.2.3 that

$$\epsilon ne_1 + \sigma(\psi(R)) - \psi(y) + \{\psi(z), \alpha\} = \{0, f_2 - f_1\},$$

where  $\epsilon = \epsilon(z, y) = \epsilon(z', y)$  (in view of  $\iota(z) = \iota(z') = g$ ) and  $R$  is the same fixed subsequence of  $W_0^{(1)}$  used in both swaps (also possible since  $\iota(z) = \iota(z') = g$ ). Hence  $\psi(z) - \alpha = \pm(f_2 - f_1)$ , contradicting that  $\alpha - \psi(z) \in \langle f_1 \rangle$ , and completing CASE 1.

CASE 2:  $W_0 \notin \mathcal{C}_0$  and  $W \in \Omega_0^{nu}$ .

Then  $W_0 \in \mathcal{C}_1$  (by our normalizing assumptions). If there is  $Z \in \mathcal{D}_1 \cap \mathcal{C}_0$  and  $\mathcal{D}_1 \cap \mathcal{C}_2 = \emptyset$ , then, in view of Lemma 3.3.4, we may assume that performing any type III swap between  $z|Z$  and  $x|x_0^{-1}W_0^{(2)}$  results in a product decomposition  $W'$  with  $\sigma(W'_0) = \sigma(W_0)$ , else CASE 1 applied to  $W'$  completes the

proof. Note that Lemma 3.3.1 guarantees the same for any  $Z \in \mathcal{D}_1 \cap \mathcal{C}_1$ . Thus if  $\mathcal{D}_1 \cap \mathcal{C}_2 \neq \emptyset$ , then (a) holds, contrary to assumption, and so we may assume instead that  $\mathcal{D}_1 \cap \mathcal{C}_2 \neq \emptyset$ .

Suppose there is  $Z \in \mathcal{D}_2$  with  $\sigma(Z) = f_1 + f_2$ . Then performing type II swaps between some  $z|Z$  and each  $x|x_0^{-1}W_0^{(2)}$  (using the same  $R|W_0^{(1)}$  for every swap, which is possible since  $\iota(x) = g$  for all  $x|x_0^{-1}W_0^{(2)}$ ), we conclude from Lemma 3.2.4 that  $\psi_1$  is constant on  $x_0^{-1}W_0^{(2)}$ . If we perform type III swaps between  $U$  and  $W_0$  with  $U \in \mathcal{D}_1 \cap \mathcal{C}_2$ , then we conclude from Lemmas 3.2.3 and 3.4.3 that there is  $u_0|x_0^{-1}W_0^{(2)}U$  such that  $\psi(x) = \alpha$  (say) for all  $x|u_0^{-1}x_0^{-1}W_0^{(2)}U$  and  $\psi(u_0) = \alpha$  or  $\alpha \pm (f_2 - f_1)$ . Thus, as  $\psi_1$  is constant on  $x_0^{-1}W_0^{(2)}$ , we conclude that  $\psi(x) = \alpha$  for all  $x|x_0^{-1}W_0^{(2)}$ . If  $u_0|U$  with  $\psi(u_0) = \alpha + f_2 - f_1$ , then swapping  $u_0|U$  for  $x|x_0^{-1}W_0^{(2)}$  results in a new product decomposition  $W'$  such that  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$ ,  $\sigma(W'_0) = f_2$ , and  $\psi_2$  is not constant on  $x_0^{-1}W_0'^{(2)}$ . However repeating the argument from the beginning of the paragraph for  $W'$ , using Lemma 3.2.5 in place of Lemma 3.2.4, we see that  $\psi_2$  must be constant on  $x_0^{-1}W_0'^{(2)}$ , a contradiction. Thus we see that any type III swap between  $u|U \in \mathcal{D}_1 \cap \mathcal{C}_2$  and  $x|x_0^{-1}W_0^{(2)}$  results in a product decomposition  $W'$  with  $\sigma(W'_0) = \sigma(W_0)$ . As a result, since  $Z \in \mathcal{D}_2$  with  $\sigma(Z) = f_1 + f_2$ , it follows from Lemma 3.3.1 that (a) holds, contrary to assumption. So we may assume  $\mathcal{D}_2 \cap \mathcal{C}_0$  is empty. Thus, in view of  $\mathcal{D}_1 \cap \mathcal{C}_2 \neq \emptyset$  and the hypotheses, it follows that there is  $U \in \mathcal{D}_2 \cap \mathcal{C}_1$ .

Performing type II swaps between some  $y|U$  and each  $x|x_0^{-1}W_0^{(2)}$  (using the same  $R|W_0^{(1)}$  for every swap), we conclude from Lemma 3.2.1 that  $\psi_1$  is constant on  $x_0^{-1}W_0^{(2)}$ . Consequently, performing type III swaps between  $W_0$  and each  $V_i \in \mathcal{D}_1 \cap \mathcal{C}_2$ , we conclude from Lemmas 3.2.3 and 3.4.3 that there exists  $v_i|V_i$  such that  $\psi(x) = \alpha$  (say) for all  $x|v_i^{-1}x_0^{-1}W_0^{(2)}V_i$ ; moreover,  $\psi(v_i) = \alpha$  or  $\alpha + f_2 - f_1$ . If there is  $Z \in \mathcal{D}_1 \cap \mathcal{C}_0$ , then, performing type III swaps between the  $x|x_0^{-1}W_0^{(2)}$  and  $z|Z$ , and between the  $x|V_i \in \mathcal{D}_1 \cap \mathcal{C}_2$  and  $z|Z$ , we conclude from Lemmas 3.3.4 and 3.3.5 that  $\psi(x) = \alpha$  for all  $x|Z$ .

If  $Z \in \mathcal{D}_1 \cap \mathcal{C}_0$  does not exist, then  $|\mathcal{D}_1| \geq 2$  and  $|\mathcal{D}_2 \cap \mathcal{C}_1| \geq 1$  ensure  $|\mathcal{D}_1 \cap \mathcal{C}_2| \geq 2$ , and, performing type III swaps between the  $V \in \mathcal{D}_1 \cap \mathcal{C}_2$ , we conclude from Lemma 3.3.2 that  $\psi(x) = \alpha$  for all  $x|V$  with  $V \in \mathcal{D}_1 \cap \mathcal{C}_2$ , completing the proof. On the other hand, if there is  $Z \in \mathcal{D}_1 \cap \mathcal{C}_0$ , then applying type III swaps between  $Z$  and each  $V_i \in \mathcal{D}_1 \cap \mathcal{C}_2$ , we conclude from Lemma 3.2.5 that  $\psi_2$  is constant on  $V_i$  and  $Z$ ; consequently, since  $\psi(v_i) = \alpha$  or  $\alpha + f_2 - f_1$ , and since  $\psi(v) = \alpha$  for all  $v|v_i^{-1}V_i$ , we conclude that  $\psi(v_i) = \alpha$  as well, completing the proof.

CASE 3:  $W_0 \notin \mathcal{C}_0$  and  $W \in \Omega_0^u$ .

Then  $W_0 \in \mathcal{C}_1$  and  $\mathcal{D}_1 \subset \mathcal{C}_0$  (else (a) holds in view of Lemma 3.1.1). Since  $|\mathcal{D}_2| \geq 1$ , there is  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$ . Performing type II swaps between each  $x|x_0^{-1}W_0$  and some fixed  $u|U$  (using the same fixed sequence  $R|W_0^{(1)}$  in each swap), it follows from Lemma 3.1.1 that  $\psi(x) = \alpha$  (say) for all  $x|x_0^{-1}W_0^{(2)}$ . Let  $V_i \in \mathcal{D}_1$ . Performing type III swaps between  $W_0$  and  $V_i$ , we conclude from Lemmas 3.1.2 and 3.4.3 that  $\psi(z) = \alpha$  for all  $z|v_i^{-1}V_i$ , for some  $v_i|V_i$ ; moreover, either  $\psi(v_i) = \alpha$  or  $\psi(v_i) = \alpha - \sigma(W_0) + \sigma(V_i)$ . However, in the latter case, since  $V_i \in \mathcal{C}_0$  and  $W_0 \in \mathcal{C}_1$  (so that  $\sigma(W_0) = f_1$  and  $\sigma(V_i) = Cf_1 + f_2$ , for some  $C \in \mathbb{Z}$ ), we see that  $\psi_2(v_i) \neq \psi_2(\alpha)$ . Since  $|\mathcal{D}_1| \geq 2$ , performing type III swaps between the  $V_i \in \mathcal{D}_1$ , we conclude from Lemma 3.1.3 that  $\psi_2$  is constant on each  $V_i$ , whence  $\psi_2(v_i) \neq \psi_2(\alpha)$  is impossible. Thus  $\psi(z) = \alpha$  for all  $z|V_i$  with  $V_i \in \mathcal{D}_1$ , completing the proof.  $\square$

Lemma 5.3 allows us to conclude detailed information concerning the values of  $\psi$  on  $W_0^{(1)}$ . Depending on  $\sigma(W_j)$  and  $\sigma(W_0)$ , the appropriate part of Lemma 3.1, 3.2 or 3.3 will ensure that one of the hypotheses in 1, 2, or 3 holds.

**Lemma 5.3.** *Let  $W \in \Omega_0$  and  $W_j \in \mathcal{A}_2^*$  be such that there are  $Y|W_j$  and  $X|W_0^{(2)}$  with  $|X| = |Y|$  and  $\epsilon'(X, Y) \notin \{1, n\}$ , and set*

$$\mathcal{D} = \{W' \in \Omega' \mid W' \text{ is the result of performing a type II swap between } X|W_0 \text{ and } Y|W_j\}.$$

1. *If  $\sigma(W'_j) - \sigma(W_j) = 0$ , for all  $W' \in \mathcal{D}$ , then  $|\text{supp}(\psi(W_0^{(1)}))| = 1$ .*
2. *If  $\sigma(W'_j) - \sigma(W_j) \in \langle f_i \rangle$ , where  $i \in \{1, 2\}$ , for all  $W' \in \mathcal{D}$ , then  $|\text{supp}(\psi_{3-i}(W_0^{(1)}))| = 1$ .*
3. *If  $\sigma(W'_j) - \sigma(W_j) \in \{0, F\}$ , for all  $W' \in \mathcal{D}$ , where  $F \in \text{Ker}(\varphi)$ , then  $\text{supp}(\psi(W_0^{(1)})) = \{\gamma, \beta\}$  for some  $\gamma, \beta \in \text{Ker}(\varphi)$  with  $\gamma - \beta \in \{0, \pm F\}$ .*

*Proof.* 1. By hypothesis, there is only one possibility for  $\sigma(\psi(R))$ , where  $R|W_0^{(1)}$  is any subsequence with  $|R| = n - \epsilon'(X, Y)$ . Furthermore, we have  $1 \leq |R| \leq n - 2 < |\psi(W_0^{(1)})|$ , and thus 1 follows from Lemma 3.5.3 applied to  $\psi(W_0^{(1)})$ .

2. The argument is analogous to that of item 1, using the group  $\text{Ker}(\varphi)/\langle f_i \rangle \cong \langle f_{3-i} \rangle$  in place of  $\text{Ker}(\varphi)$ .

3. By the arguments for item 1, replacing Lemma 3.5.3 by Lemma 3.5.1, we conclude that  $\psi(W_0^{(1)}) = \gamma^l \beta^{n-1-l}$  (say), where  $l \geq n - 1 - l \geq 1$  and  $\gamma \neq \beta$  (else the lemma is complete); moreover,

$$\epsilon(X, Y)ne_1 + \sigma(\psi(X)) - \sigma(\psi(Y)) + \min\{t, l\} \cdot \gamma + (t - \min\{t, l\}) \cdot \beta + \{0, \beta - \gamma\} = \{0, F\},$$

where  $t = n - \epsilon'(X, Y)$ . Thus  $\beta - \gamma = \pm F$ , as desired.  $\square$

The following lemma encapsulates an alignment argument for the  $\iota$  values that forces them to live in near disjoint intervals. It will be a key part of the more difficult portions of CLAIM C.

**Lemma 5.4.** *Let  $W \in \Omega_0$ , let  $\mathcal{D} \subset \mathcal{A}_2^*$  be nonempty, and let  $Z|W_0^{(2)}$  be nontrivial. For  $x|S$ , let  $\psi_0(x) = \psi(x)$ , and for  $x \in \text{Ker}(\varphi)$ , let  $\psi_0$  be the identity map. Let  $i \in \{0, 1, 2\}$ . If  $\psi_i(ne_1) \neq 0$  and*

$$(7) \quad \psi_i(x) - \psi_i(y) + \psi_i(\epsilon(x, y)ne_1) = 0$$

*for every  $x|Z$  and  $y|U \in \mathcal{D}$ , then there exist intervals  $J_1, J_2$  and  $J_3$  of  $\mathbb{Z}$  with either*

$$(8) \quad \text{supp}(\iota(\prod_{U \in \mathcal{D}} U)) \subset J_3, \text{supp}(\iota(Z)) \subset J_1 \cup J_2, \quad \text{and} \quad \max J_1 \leq \min J_3 \leq \max J_3 < \min J_2, \quad \text{or}$$

$$(9) \quad \text{supp}(\iota(Z)) \subset J_3, \text{supp}(\iota(\prod_{U \in \mathcal{D}} U)) \subset J_1 \cup J_2, \quad \text{and} \quad \max J_1 < \min J_3 \leq \max J_3 \leq \min J_2.$$

*Moreover,  $I$  can be chosen such that:*

1.  *$\min I$  is congruent to an element in  $\iota(Z)$  modulo  $n$ ,*
2.  *$\iota(x) \leq \iota(y)$  and  $\epsilon(x, y) = 0$  for all  $x|Z$  and  $y|U \in \mathcal{D}$ , and*
3.  *$\psi_i(x) = \psi_i(y)$  for all  $xy|Z \prod_{U \in \mathcal{D}} U$ .*

*Proof.* Observe, for  $xy|S_2$ , that

$$(10) \quad \epsilon(x, y) = \begin{cases} 0, & \iota(x) \leq \iota(y); \\ 1, & \iota(x) > \iota(y). \end{cases}$$

Consequently, we conclude from (7) that

$$(11) \quad \psi_i(x) = \psi_i(y),$$

for all  $x|Z$  and  $y|U \in \mathcal{D}$  with  $\iota(x) \leq \iota(y)$ , and that

$$(12) \quad \psi_i(x) = \psi_i(y) - \psi_i(ne_1),$$

for all  $x|Z$  and  $y|U \in \mathcal{D}$  with  $\iota(x) > \iota(y)$ .

If there do not exist  $x|Z$  and  $yy'| \prod_{U \in \mathcal{D}} U$  with  $\iota(x) \leq \iota(y)$  and  $\iota(x) > \iota(y')$ , then, for every  $x|Z$ , we have either  $\iota(x) \leq \iota(y)$  for all  $y| \prod_{U \in \mathcal{D}} U$ , or  $\iota(x) > \iota(y)$  for all  $y| \prod_{U \in \mathcal{D}} U$ . Thus we see that (8) holds (with  $J_3 = [\min(\text{supp}(\iota(\prod_{U \in \mathcal{D}} U))), \max(\text{supp}(\iota(\prod_{U \in \mathcal{D}} U)))]$ ,  $J_1$  being any nonempty interval containing those  $\iota(x)$  with  $\iota(x) \leq \iota(y)$  for all  $y| \prod_{U \in \mathcal{D}} U$  and  $\max J_1 \leq \min J_3$ , and  $J_2$  being any nonempty interval containing those  $\iota(x)$  with  $\iota(x) > \iota(y)$  for all  $y| \prod_{U \in \mathcal{D}} U$  and  $\min J_2 > \max J_3$ ).

Now instead let  $x|Z$  and  $yy'| \prod_{U \in \mathcal{D}} U$  with  $\iota(x) \leq \iota(y)$  and  $\iota(x) > \iota(y')$ , and factor  $\prod_{U \in \mathcal{D}} U = J'_1 J'_2$ , where  $J'_1$  are those terms  $a| \prod_{U \in \mathcal{D}} U$  with  $\iota(a) < \iota(x)$ , and  $J'_2$  are those terms  $b| \prod_{U \in \mathcal{D}} U$  with  $\iota(b) \geq \iota(x)$ . By assumption, both  $J'_i$  are nontrivial. Moreover, from (11) and (12) and  $\psi_i(ne_1) \neq 0$ , we see that

$$(13) \quad \psi_i(b) = \psi_i(x)$$

and

$$(14) \quad \psi_i(a) = \psi_i(x) + \psi_i(ne_1) \neq \psi_i(x),$$

for all  $a|J'_1$  and  $b|J'_2$ . Thus  $\psi_i$  is constant on  $J'_1$  and also on  $J'_2$  but the two values assumed are distinct. If there were  $x'|Z$  such that  $\iota(x') \leq \max(\text{supp}(\iota(J'_1)))$ , then by (11) and (13) we would conclude that  $\psi_i(x') = \psi_i(b) = \psi_i(x)$ , where  $b$  is any term of  $J'_2$ , while by applying (11) and (14) between  $x'$  and  $\max(\text{supp}(\iota(J'_1))) := a_0$ , we would conclude that  $\psi_i(x') = \psi_i(a_0) = \psi_i(x) + \psi_i(ne_1) \neq \psi_i(x)$ , a contradiction to what we have just seen. We likewise obtain a contradiction if there were  $x'|Z$  such that  $\iota(x') > \min(\text{supp}(\iota(J'_2)))$ . Therefore we see that (9) holds with  $J_1 = [\min(\text{supp}(\iota(J'_1))), \max(\text{supp}(\iota(J'_1)))]$ ,  $J_2 = [\min(\text{supp}(\iota(J'_2))), \max(\text{supp}(\iota(J'_2)))]$ , and  $J_3 = [\min(\text{supp}(\iota(Z))), \max(\text{supp}(\iota(Z)))]$ .

Choosing  $I$  such that  $\min I$  is congruent to  $\min(\text{supp}(\iota(Z)))$  modulo  $n$ , if either (9) holds or else (8) holds with  $\text{supp}(\iota(Z)) \cap J_2 = \emptyset$ , and congruent to  $\min(\text{supp}(\iota(Z)) \cap J_2)$  otherwise, the remaining properties follow in view of (7) and (10).  $\square$

Now we choose a product decomposition  $W \in \Omega_0$ , and if  $\Omega_0^u \neq \emptyset$ , we assume that  $W \in \Omega_0^u$ .

**CLAIM A:**  $h(S_1) \geq |S_1| - 1$ .

*Proof.* We need to show that there exists  $x_0|S_1$  such that  $\psi(x) = \psi(y)$  for all  $xy|x_0^{-1}S_1$ . We divide the proof into four main cases. In many of the cases, we obtain partial works towards showing  $h(S_1) = |S_1|$ , which will later be utilized in CLAIM B.

CASE 1:  $\Omega_0^u \neq \emptyset$ ,  $|\mathcal{A}_1| \geq 2$  and  $|\mathcal{C}_1 \cap \mathcal{A}_1| \geq 1$ .

In this case, we will moreover show that  $h(S_1) = |S_1|$  unless  $|\mathcal{A}_1 \cap \mathcal{C}_0| = 1$  or  $|\mathcal{A}_1 \cap \mathcal{C}_1| = 1$ , and that  $|\text{supp}(\psi(U))| > 1$  for  $U \in \mathcal{A}_1 \cap \mathcal{C}_i$ , where  $i \in \{1, 2\}$ , is only possible when  $|\mathcal{A}_1 \cap \mathcal{C}_i| = 1$ .

If  $U, V \in \mathcal{A}_1$  are distinct, then we can perform a type I swap between  $U$  and  $V$ , and by (4) and Lemma 3.1, we conclude that

$$(15) \quad \begin{aligned} \sigma(\psi(X)) - \sigma(\psi(Y)) &= 0, & \text{if } U, V \in \mathcal{C}_1 \\ \sigma(\psi(X)) - \sigma(\psi(Y)) &\in \{0, (1-C)f_1 - f_2\}, & \text{if } U \in \mathcal{C}_1, V \in \mathcal{C}_0 \text{ and } \sigma(V) = Cf_1 + f_2 \\ \sigma(\psi(X)) - \sigma(\psi(Y)) &\in \langle f_1 \rangle, & \text{if } U, V \in \mathcal{C}_0, \end{aligned}$$

for  $X|U$  and  $Y|V$  with  $|X| = |Y|$ .

If  $|\mathcal{A}_1 \cap \mathcal{C}_0| \geq 2$ , then using (15) (running over all  $X$  and  $Y$  with  $|X| = |Y| = 1$ ), we conclude that  $\psi(x) - \psi(y) \in \langle f_1 \rangle$  for all  $x$  and  $y$  dividing a block from  $\mathcal{A}_1 \cap \mathcal{C}_0$ .

If  $|\mathcal{A}_1 \cap \mathcal{C}_1| \geq 2$ , then using (15) (running over all  $X$  and  $Y$  with  $|X| = |Y| = 1$ ) and Lemma 3.4.1, we conclude that  $\psi(x) = \psi(y)$  for all  $x$  and  $y$  dividing a block from  $\mathcal{A}_1 \cap \mathcal{C}_1$ .

If  $U \in \mathcal{A}_1 \cap \mathcal{C}_1$  and  $V \in \mathcal{A}_1 \cap \mathcal{C}_0$  with  $U$  and  $V$  distinct, then, using (15) (running over all  $X$  and  $Y$  with  $|X| = |Y| \leq 2 \leq n-1$ ) and Lemma 3.4.3, we conclude that  $\psi(x) = \alpha$  (say) for all  $x|x_0^{-1}UV$ , for some  $x_0|UV$ ; moreover,  $\psi(x_0) = \alpha$  or  $\alpha \pm ((1-C)f_1 - f_2)$ .

Suppose  $x_0|U$  and  $\psi(x_0) \neq \alpha$ . Then in view of the fourth paragraph of CASE 1, we see that  $|\mathcal{A}_1 \cap \mathcal{C}_1| = 1$ . Thus performing type I swaps between  $U$  and all possible  $V \in \mathcal{A}_1 \cap \mathcal{C}_0$  completes CLAIM A, for  $n \geq 5$  or  $U \neq W_0$ , and, when  $n = 3$  and  $U = W_0$ , we instead conclude that either  $\psi(V) = \alpha^n$  or  $\psi(V) = \beta^n$ , where  $\psi(W_0^{(1)}) = \alpha\beta$ , for all  $V \in \mathcal{A}_1 \cap \mathcal{C}_0$ . However, if there are  $V, V' \in \mathcal{A}_1 \cap \mathcal{C}_0$  with  $\psi(V) = \alpha^n$  and  $\psi(V') = \beta^n$  and  $\alpha \neq \beta$ , then (15) implies that  $\beta - \alpha = (1-C)f_1 - f_2$  and  $\alpha - \beta = (1-C')f_1 - f_2$ , where  $\sigma(V) = Cf_1 + f_2$  and  $\sigma(V') = C'f_1 + f_2$ , from which we conclude that  $(2-C'-C)f_1 - 2f_2 = 0$ , contradicting that  $m \geq 3$ . So we may instead assume  $x_0|V$ .

In this case, in combination with the results of the previous paragraphs, we find that there is at most one  $v_i|V_i$ , for each  $V_i \in \mathcal{A}_1 \cap \mathcal{C}_0$ , such that  $\psi(x) = \alpha$  for all  $x|S_1$  not equal to any  $v_i$ . In this scenario, CLAIM A is done unless we have two distinct  $V_1, V_2 \in \mathcal{A}_1 \cap \mathcal{C}_0$  such that  $\psi(v_1) \neq \alpha$  and  $\psi(x) = \alpha$  for all  $x|v_1^{-1}v_2^{-1}UV_1V_2$ . However, applying a type I swap between  $y|U$  and  $v_1|V_1$ , we conclude from (15) that  $\alpha - \psi(v_1) = (1-C)f_1 - f_2 \notin \langle f_1 \rangle$ , for some  $C \in \mathbb{Z}$ , which, in view of  $\alpha\psi(v_1)|\psi(V_1)$ , contradicts the conclusion of the third paragraph of CASE 1. This completes CASE 1.

CASE 2:  $|\mathcal{A}_1| = 1$ .

In this case, we will show that  $h(S_1) = |S_1|$ .

Suppose  $W_0 \in \mathcal{C}_0$ . Then we may choose  $f_2$  such that  $\sigma(W_0) = f_1 + f_2$ , and if  $\Omega_0^u = \emptyset$ , such that  $\tilde{\sigma}(W) = f_1^{m-1}f_2^{m-1}(f_1 + f_2)$  also. Let  $\mathcal{D}_1$  be those blocks  $W_i$  with  $\sigma(W_i) = f_1$  and let  $\mathcal{D}_2$  be all other blocks from  $\mathcal{A}_2^*$ . Applying Lemma 5.2, we see that Lemma 5.2(ii) must hold, else  $ge_1 + e_2 + \alpha$  will have multiplicity at least  $mn - 1$  in  $S$ , as desired. Performing a type II swap between the  $X|W_0$  and  $Y|W_j$  given by Lemma 5.2(ii), we conclude, from Lemmas 5.3.2 and either 3.1.2 (since  $\sigma(W_0) = f_1 + f_2$ ) or 3.2.4, that  $\psi_1$  is constant on  $W_0^{(1)}$ . However, reversing the roles of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  and repeating the above argument using Lemmas 3.1.3 and 3.2.5 in place of Lemmas 3.1.2 and 3.2.4, we conclude that  $\psi_2$  is also constant on  $W_0^{(1)}$ , whence  $\psi$  is constant on  $W_0^{(1)}$ , completing the proof of CLAIM A. So we may assume  $W_0 \notin \mathcal{C}_0$ .

Suppose  $\Omega_0^u = \emptyset$ . Then we may w.l.o.g. assume  $\tilde{\sigma}(W) = f_1^{m-1}f_2^{m-1}(f_1 + f_2)$ , that  $\mathcal{C}_1$  consists of those blocks  $W_i$  with  $\sigma(W_i) = f_1$ , and that  $\sigma(W_0) = f_1$ . Let  $\mathcal{D}_1 = \mathcal{C}_2$  and  $\mathcal{D}_2 = \mathcal{C}_1^* \cup \mathcal{C}_0$ . Applying Lemma 5.2, we see that Lemma 5.2(ii) must hold, else there will be a term with multiplicity at least  $mn - 1$  in  $S$ , as desired. Thus Lemmas 5.3.3 and 3.2.3 imply that  $\text{supp}(\psi(W_0^{(1)})) = \{\gamma, \beta\}$  (say) with  $\beta - \gamma = \pm(f_2 - f_1)$  (else CLAIM A follows).

Reversing the roles of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  and again applying Lemma 5.2, we once more see that Lemma 5.2(ii) must hold, else there is a term with multiplicity  $mn - 1$  in  $S$ , as desired. Thus Lemma 5.3.2 and either Lemma 3.2.1 or 3.2.4 imply that  $\psi_1$  is constant on  $W_0^{(1)}$ , contradicting that  $\beta - \gamma = \pm(f_2 - f_1)$ . So we may assume  $\Omega_0^u \neq \emptyset$ .

Let w.l.o.g.  $W_1, \dots, W_{m-2}$  be the blocks of  $\mathcal{C}_1^* \cap \mathcal{A}_2$ , and let  $\mathcal{D}_1 = \mathcal{C}_1^*$  and  $\mathcal{D}_2 = \mathcal{C}_0$ . Apply Lemma 5.2. If Lemma 5.2(ii) holds, then Lemmas 5.3.1 and 3.1.1 imply that  $\psi$  is constant on  $W_0^{(1)}$ , whence CLAIM A follows. Therefore we may instead assume  $\iota(x) = g$  and  $\psi(x) = \alpha$  (say) for all terms  $x|x_0^{-1}W_0^{(2)}W_1 \dots W_{m-2}$ , for some  $x_0|W_0^{(2)}$  with  $\iota(x_0) \equiv g+1 \pmod n$ .

Consider  $W_j$  with  $j \geq m-1$ . If  $\iota(W_j) \neq g^n$ , then there exist  $x|W_0^{(2)}$  and  $y|W_j$  with  $\epsilon'(x, y) \notin \{1, n\}$ , whence Lemmas 5.3.3 and 3.1.2 imply that  $\text{supp}(\psi(W_0^{(1)})) = \{\gamma, \beta\}$  (say) with  $\beta - \gamma = \pm F_j$  (else CLAIM A follows), where  $F_j = (1 - C_j)f_1 - f_2$  and  $\sigma(W_j) = C_j f_1 + f_2$ .

If  $W_k$  is another block with  $k \geq m-1$  and  $\iota(W_k) \neq g^n$ , then the above paragraph implies that  $\beta - \gamma = \pm F_k$ , where  $F_k = (1 - C_k)f_1 - f_2$  and  $\sigma(W_k) = C_k f_1 + f_2$ . Thus, since  $m \geq 3$  and  $\beta - \gamma = \pm F_j$ , we conclude that  $F_j = F_k$  and  $C_j \equiv C_k \pmod m$ . As a result, we see that any two blocks  $W_j$  and  $W_k$ , with  $j, k \geq m-1$  and  $\iota(W_j), \iota(W_k) \neq g^n$ , must have  $\sigma(W_j) = \sigma(W_k)$ . Hence, since  $W \in \Omega_0^u$ , we conclude that there are at least two distinct blocks  $W_s$  and  $W_r$  with  $s, r \geq m-1$  and  $\iota(W_s) = \iota(W_r) = g^n$ . Performing type III swaps between  $W_0$  and both  $W_s$  and  $W_r$ , we conclude from Lemmas 3.1.2 and 3.4.3 that  $\psi(x) = \alpha$  for all but at most two terms of  $W_s W_r$ , whence  $ge_1 + e_2 + \alpha$  has multiplicity at least  $(m-1)n - 1 + 2n - 2 \geq mn$  in  $S$ , contradicting that  $S \in \mathcal{A}(G)$  and completing CASE 2.

CASE 3:  $\Omega_0^u \neq \emptyset$ ,  $|\mathcal{A}_1| \geq 2$  and  $|\mathcal{C}_1 \cap \mathcal{A}_1| = 0$ .

In this case, we will moreover show that  $h(S_1) = |S_1|$ .

We may w.l.o.g. assume  $W_1, \dots, W_{m-1}$  are the blocks in  $\mathcal{C}_1 \cap \mathcal{A}_2$ . Let  $\mathcal{D}_1 = \mathcal{C}_1$  and  $\mathcal{D}_2 = \mathcal{C}_0^* \cap \mathcal{A}_2$ . If  $|\mathcal{D}_2| \geq 1$ , then we can apply Lemma 5.2. Otherwise, in view of Lemma 3.1.2, we may assume hypothesis (a) holds in Lemma 5.2, else applying CASE 1 to the resulting product decomposition  $W'$  would imply, in view of  $|\mathcal{D}_2| = 0$ , that  $\psi(x) = \alpha$  (say) for all  $x|W'_i = W_i$  with  $i \in [m, 2m-2]$ , in which case  $\sigma(W'_i) = ne_1 + n\alpha$  has multiplicity  $m-1$  in  $\tilde{\sigma}(W')$ , contradicting that  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$  (in view of Lemma 3.1.2) with  $W \in \Omega_0^u$ . Thus, in either case Lemma 5.2 is available. If Lemma 5.2(i) holds, then  $ge_1 + e_2 + \alpha$  is a term with multiplicity at least  $mn - 1$  in  $S$ , as desired. Therefore there is  $X|W_0^{(2)}$  and  $Y|W_j$ , for some  $j \in [1, m-1]$ , such that  $|X| = |Y|$  and  $\epsilon'(X, Y) \notin \{1, n\}$ . Hence Lemmas 5.3.3 and 3.1.2 imply that  $\text{supp}(\psi(W_0^{(1)})) = \{\gamma, \beta\}$  (say) with  $\gamma - \beta \in \{0, \pm F\}$ , where  $F = (C-1)f_1 + f_2$  and  $\sigma(W_0) = C f_1 + f_2$ . Since  $|\mathcal{A}_1| \geq 2$ , let  $V \in \mathcal{C}_0^* \cap \mathcal{A}_1$ . Performing type I swaps between  $W_0$  and  $V$ , we conclude from Lemma 3.1.3 that  $\psi_2$  is constant on  $VW_0^{(1)}$ , whence  $\gamma - \beta \in \{0, \pm F\}$  implies  $\gamma = \beta$ .

Performing type I swaps among the  $V \in \mathcal{C}_0 \cap \mathcal{A}_1$ , we conclude from Lemma 3.1.3 that  $\psi_2(x) = \psi_2(\gamma)$  for all  $x|V \in \mathcal{C}_0 \cap \mathcal{A}_1$ . Let  $W'$  be the product decomposition resulting from performing a type II swap between  $X|W_0$  and  $Y|W_j$  (with  $X$  and  $Y$  as given by Lemma 5.2(ii) in the previous paragraph). Since  $\epsilon'(X, Y) \notin \{1, n\}$ , we conclude that there is a block  $W'_k \in \mathcal{C}_1$ , with  $k \in \{0, j\}$ , having  $(e_1 + \gamma)|W'_k$ . Since  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$  (in view of Lemma 3.1.2), performing type I swaps between  $W'_k$  and each distinct block  $V' = V \in \mathcal{C}_0^* \cap \mathcal{A}_1$ , we conclude from Lemma 3.1.2 that either  $\psi(x) = \gamma$  or  $\psi(x) = \gamma + \sigma(V') - \sigma(W'_k)$ , for each  $x|V'$ . However, since  $W'_k \in \mathcal{C}_1$  and  $V' \in \mathcal{C}_0$ , it follows that the latter contradicts that  $\psi_2$  is constant on  $V|W_0^{(1)}$  with value  $\psi_2(\gamma)$ . Therefore we conclude that  $\psi(x) = \gamma$  for all  $x|V'$ , with  $V' = V^* \in \mathcal{C}_0 \cap \mathcal{A}_1$ , whence  $\psi(x) = \gamma$  for all  $x|S_1$ , as desired, completing CASE 3.

CASE 4:  $\Omega_0^u = \emptyset$  and  $|\mathcal{A}_1| \geq 2$ .

We may w.l.o.g. assume  $\tilde{\sigma}(W) = f_1^{m-1} f_2^{m-1} (f_1 + f_2)$ , by an appropriate choice of  $f_2$ , whence CLAIM A follows easily by performing type I swaps between the blocks of  $\mathcal{A}_1$  and using Lemmas 3.3 and 3.4. This completes CASE 4.  $\square$



In view of CLAIM A, we may assume  $S_1 = e_1^{|S_1|-1}(e_1 + a)$ , for some  $a \in \text{Ker}(\varphi)$ . Let  $y_0 = e_1 + a$ .

**CLAIM B:**  $h(S_1) = |S_1|$ .

*Proof.* We assume by contradiction  $a \neq 0$ . In view of the partial conclusions of CLAIM A, we may assume  $|\mathcal{A}_1| \geq 2$  (in view of CASE 2 of CLAIM A), and, if  $\Omega_0^u \neq \emptyset$ , that  $|\mathcal{A}_1 \cap \mathcal{C}_1| \geq 1$  (in view of CASE 3 of CLAIM A). We proceed in four cases.

CASE 1:  $\Omega_0^u \neq \emptyset$  and  $y_0|U$  for some  $U \in \mathcal{A}_1 \cap \mathcal{C}_1$ .

In view of CASE 1 of CLAIM A, we have  $|\mathcal{A}_1 \cap \mathcal{C}_1| = 1$ . Hence, if  $U \neq W_0$ , then  $W_0 \in \mathcal{C}_0$ , and performing a type I swap between  $y_0|U$  and some  $y|W_0$  results (in view of Lemma 3.1.2) in a new product decomposition  $W'$  with  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$ ,  $U' \in \mathcal{C}_0$ ,  $W'_0 \in \mathcal{C}_1$ ,  $y_0|W'_0$  and  $W'$  also satisfying the hypothesis of CASE 1. On the other hand, if  $U = W_0$ , then  $|\mathcal{A}_1| \geq 2$  and  $|\mathcal{A}_1 \cap \mathcal{C}_1| = 1$  imply that there is  $V \in \mathcal{A}_1^* \cap \mathcal{C}_0$ , and performing a type I swap between  $y_0|W_0$  and some  $y|V$  results (in view of Lemma 3.1.2) in a new product decomposition  $W'$  with  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$ ,  $W'_0 \in \mathcal{C}_0$ ,  $V' \in \mathcal{C}_1$ ,  $y_0|V'$  and  $W'$  also satisfying the hypothesis of CASE 1. Thus w.l.o.g. we may assume  $U \neq W_0$ . Since  $U \in \mathcal{C}_1$  and  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$  with  $W'_0 \in \mathcal{C}_1$  (with  $W'$  as in the second sentence of CASE 1), then, letting  $\sigma(W_0) = Cf_1 + f_2$ , we see that  $a = (1 - C)f_1 - f_2$ .

Let  $\mathcal{D}_1 = \mathcal{A}_2^*(W') \cap \mathcal{C}_1(W')$  and  $\mathcal{D}_2 = \mathcal{A}_2^*(W') \cap \mathcal{C}_0(W')$ . Since  $|\mathcal{A}_1 \cap \mathcal{C}_1| = 1$  and  $W'_0 \in \mathcal{C}_1$ , we have  $|\mathcal{D}_1| = m - 2$ , and by CLAIM A we have  $|\mathcal{D}_2| \geq 1$  (else  $e_1$  is a term with multiplicity at least  $(m + 1)n - 2 \geq mn$ , contradicting that  $S \in \mathcal{A}(G)$ ). If Lemma 5.2(ii) holds for  $W'$ , then Lemmas 5.3.1 and 3.1.1 imply that  $a = 0$ , a contradiction. Therefore Lemma 5.2(i) holds for  $W'$ . Let  $g$  and  $\alpha$  be as given by Lemma 5.2(i).

Since  $|\mathcal{D}_2| \geq 1$ , let  $V \in \mathcal{A}_2^*(W) \cap \mathcal{C}_0(W)$ . If  $\iota(V) = g^n$ , then, performing type III swaps between  $V$  and some  $Z \in \mathcal{A}_2^* \cap \mathcal{C}_1$ , and between  $V$  and  $W_0$ , we conclude from Lemmas 3.1.2, 3.1.3 and 3.4.3 that  $\psi(x) = \alpha$  for all  $x|V$ , whence  $ge_1 + e_2 + \alpha$  has multiplicity at least  $mn - 1$  in  $S$ , as desired. Therefore, in view of  $\iota(W_0^{(2)}) \equiv g^{n-1}(g + 1) \pmod{n}$ , we see that there exists  $x|W_0^{(2)} = W_0'^{(2)}$  and  $y|V = V'$  such that  $e'(x, y) \notin \{1, n\}$ . Hence, from Lemmas 5.3.3 (applied to  $W'$ ) and 3.1.2, it follows that  $a = \pm((1 - C')f_1 - f_2)$ , where  $\sigma(V) = C'f_1 + f_2$ . Thus, since  $a = (1 - C)f_1 - f_2$  and  $m \geq 3$ , we conclude that  $C'f_1 = Cf_1$  and  $\sigma(V) = \sigma(W_0)$ . As  $V \in \mathcal{A}_2^*(W) \cap \mathcal{C}_0(W)$  was arbitrary, we see that  $\sigma(V) = Cf_1 + f_2$  for all  $V \in \mathcal{A}_2(W) \cap \mathcal{C}_0(W)$ . On the other hand, if  $Z \in \mathcal{A}_1(W) \cap \mathcal{C}_0(W)$ , then, performing type I swaps between  $U$  and  $Z$ , we conclude from Lemma 3.1.2 that  $a = (1 - C'')f_1 - f_2$ , where  $\sigma(Z) = C''f_1 + f_2$ . Thus  $a = (1 - C)f_1 - f_2$  implies that  $C''f_1 = Cf_1$ , and now  $\sigma(Z) = Cf_1 + f_2$  for all  $Z \in \mathcal{A}_1(W) \cap \mathcal{C}_0(W)$ . Consequently,  $\sigma(Z) = Cf_1 + f_2$  for all  $Z \in \mathcal{C}_0(W)$ , contradicting that  $h(\tilde{\sigma}(W)) < m$ . This completes CASE 1.

CASE 2:  $\Omega_0^u \neq \emptyset$  and  $y_0|U$  for some  $U \in \mathcal{A}_1 \cap \mathcal{C}_0$

Recall that  $|\mathcal{A}_1 \cap \mathcal{C}_1| \geq 1$  and  $|\mathcal{A}_1| \geq 2$ . CASE 1 of CLAIM A and the hypothesis of CASE 2 further imply that  $|\mathcal{A}_1 \cap \mathcal{C}_0| = 1$ . Thus, if  $U \neq W_0$ , then  $W_0 \in \mathcal{C}_1$ , and performing a type I swap between  $y_0|U$  and some  $y|W_0$  results (in view of Lemma 3.1.2) in a product decomposition  $W'$  with  $y_0|W'_0$ ,  $W'_0 \in \mathcal{C}_0$ ,  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$  and  $W'$  satisfying the hypotheses of CASE 2. Thus w.l.o.g. we may assume  $U = W_0$ .

Since  $|\mathcal{A}_1 \cap \mathcal{C}_1| \geq 1$ , let  $V \in \mathcal{A}_1^* \cap \mathcal{C}_1$ . Performing a type I swap between  $y_0|W_0$  and some  $y|V$ , letting  $W'$  be the resulting product decomposition, we conclude from Lemma 3.1.2 that  $a = (C - 1)f_1 + f_2$ , where  $\sigma(W_0) = Cf_1 + f_2$ . Since  $|\mathcal{A}_1 \cap \mathcal{C}_0| = 1$  and  $W_0 \in \mathcal{C}_0$ , let w.l.o.g.  $W_1, \dots, W_{m-1}$  be the blocks

of  $\mathcal{A}_2^* \cap \mathcal{C}_0$ . If  $x|W_0^{(2)}$  and  $y|W_j$ , with  $j \in [1, m-1]$  and  $\iota(x) = \iota(y)$ , then, performing a type III swap between  $x|W_0$  and  $y|W_j$  and between  $x|W'_0$  and  $y|W'_j$ , we conclude in view of Lemmas 3.1.3 and 3.1.2 that  $\psi(x) = \psi(y)$ ; thus, letting  $\mathcal{D}_1 = \mathcal{A}_2^* \cap \mathcal{C}_0$  and  $\mathcal{D}_2 = \mathcal{A}_2^* \cap \mathcal{C}_1$ , we see that hypothesis (a) holds in Lemma 5.2. If Lemma 5.2(i) holds, then  $ge_1 + e_2 + \alpha$  is a term of  $S$  with multiplicity at least  $mn - 1$ , as desired. Therefore Lemma 5.2(ii) holds, whence Lemmas 5.3.2 and 3.1.3 imply that  $a \in \langle f_1 \rangle$ , contradicting that  $a = (C - 1)f_1 + f_2$ . This completes CASE 2.

Note that if  $\Omega_0^u = \emptyset$ , then (in view of  $|\mathcal{A}_1| \geq 2$ ) we may w.l.o.g. assume  $y_0|U$  with  $U \neq W_0$ , by an appropriate type I swap. Moreover, when  $\Omega_0^u = \emptyset$ , we will w.l.o.g. assume  $\tilde{\sigma}(W) = f_1^{m-1}f_2^{m-1}(f_1 + f_2)$  with  $\mathcal{C}_1$  consisting of those blocks  $W_i$  with  $\sigma(W_i) = f_1$ .

CASE 3:  $\Omega_0^u = \emptyset$  and  $y_0|U$  for some  $U \in \mathcal{A}_1^* \cap \mathcal{C}_0$ .

We may w.l.o.g. assume  $W_0 \in \mathcal{C}_1$ . Performing a type I swap between  $y_0|U$  and some  $y|W_0$ , letting  $W'$  be the resulting product decomposition, we conclude from Lemma 3.3.4 that  $a = f_2$ . Let  $\mathcal{D}_1 = \mathcal{A}_2^*(W') \cap \mathcal{C}_2(W')$  and let  $\mathcal{D}_2 = \mathcal{A}_2^*(W') \cap \mathcal{C}_1(W')$ . Observe that  $|\mathcal{D}_1| = m - 1$ , else performing a type I swap between  $y_0|U$  and some  $V \in \mathcal{A}_1 \cap \mathcal{C}_2$  would imply in view of Lemma 3.3.5 that  $a = f_1$ , contradicting that  $a = f_2$ . If a type III swap between  $W'_0$  and some  $W'_j \in \mathcal{D}_1$  results in a new product decomposition  $W''$  with  $\sigma(W''_0) \neq \sigma(W'_0)$ , then Lemma 3.3.5 implies  $\sigma(W''_0) = f_2$ , whence, performing a type I swap between  $y_0|W''_0^{(1)} = W'_0^{(1)}$  and  $U'' = U'$ , we conclude from Lemma 3.2.3 that  $-a = f_1 - f_2$ , contradicting that  $a = f_2$ . Thus hypothesis (a) of Lemma 5.2 holds for  $W'$ . If Lemma 5.2(i) holds, then  $ge_1 + e_2 + \alpha$  has multiplicity at least  $mn - 1$  in  $S$ , as desired. Therefore, Lemma 5.2(ii) holds, whence Lemmas 5.3.2 and 3.2.5 imply that  $a \in \langle f_1 \rangle$ , contradicting that  $a = f_2$  and completing CASE 3.

CASE 4:  $\Omega_0^u = \emptyset$  and  $y_0|U \in \mathcal{A}_1^*$  with  $U \notin \mathcal{C}_0$ .

We may w.l.o.g. assume  $U \in \mathcal{C}_1$ . If  $W_0 \in \mathcal{C}_1$ , then performing type I swaps between  $W_0$  and  $U$  would imply, in view of Lemma 3.3.1, that  $a = 0$ , a contradiction. Moreover, this also shows that  $\mathcal{A}_1 \cap \mathcal{C}_1 = \{U\}$ .

Suppose  $W_0 \in \mathcal{C}_2$ . Performing a type I swap between  $y_0|U$  and some  $y|W_0$ , letting  $W'$  be the resulting product decomposition, we conclude from Lemma 3.2.3 that  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$ ,  $W'_0 \in \mathcal{C}_1$ ,  $a = f_1 - f_2$  and  $ne_1 = \sigma(U') = f_2$ . Let  $\mathcal{D}_1 = \mathcal{A}_2^*(W') \cap \mathcal{C}_1(W')$  and let  $\mathcal{D}_2 = \mathcal{A}_2^*(W') \cap \mathcal{C}_0(W')$ . Since  $\mathcal{A}_1 \cap \mathcal{C}_1 = \{U\}$ , we have  $|\mathcal{D}_1| = m - 2$ . Since  $ne_1 = f_2 \neq f_1 + f_2$ , we have  $Z \in \mathcal{C}_0$  with  $Z \in \mathcal{A}_2^*$ , and thus  $|\mathcal{D}_2| \geq 1$ . Apply Lemma 5.2 to  $W'$ . If Lemma 5.2(ii) holds, then Lemmas 5.3.1 and 3.2.1 imply  $\psi_1(a) = 0$ , contradicting that  $a = f_1 - f_2$ . Therefore Lemma 5.2(i) holds, whence  $gne_1 + ne_2 + n\alpha = \sigma(V) = f_1$ , where  $V \in \mathcal{D}_1$ . If there is a type III swap between  $Z' = Z$  and  $W'_0$  resulting in a product decomposition  $W''$  with  $\sigma(W''_0) \neq \sigma(W'_0)$ , then Lemma 3.3.4 implies that  $\sigma(W''_0) = f_1 + f_2$ , whence, performing a type I swap between  $y_0|W''_0$  and  $y|U'' = U'$ , we conclude from Lemma 3.3.5 that  $-a = -f_1$ , contradicting that  $a = f_1 - f_2$ . Therefore hypothesis (a) holds in Lemma 5.2 for  $W'$  with the roles of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  reversed. Apply Lemma 5.2 in this case. If Lemma 5.2(ii) holds, then Lemmas 5.3.2 and 3.2.4 imply that  $a \in \langle f_2 \rangle$ , contradicting that  $a = f_1 - f_2$ . Therefore Lemma 5.2(i) holds, whence  $gne_1 + ne_2 + n\alpha = \sigma(Z) = f_1 + f_2$ , contradicting that  $gne_1 + ne_2 + n\alpha = f_1$ . So we may assume instead that  $W_0 \in \mathcal{C}_0$ .

Performing a type I swap between  $y_0|U$  and some  $y|W_0$ , letting  $W'$  be the resulting product decomposition, we conclude from Lemma 3.3.4 that  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$ ,  $W'_0 \in \mathcal{C}_1$ ,  $a = -f_2$ , and  $ne_1 = \sigma(U') = f_1 + f_2$ . Let  $\mathcal{D}_1 = \mathcal{A}_2^*(W') \cap \mathcal{C}_2(W')$ . If there is  $V \in \mathcal{A}_1 \cap \mathcal{C}_2$ , then, performing a type I swap between  $y_0|U$  and some  $y|V$ , we conclude from Lemma 3.2.3 that  $a = f_1 - f_2$ , contradicting that  $a = -f_2$ . Therefore  $|\mathcal{D}_1| = m - 1$ . Let  $\mathcal{D}_2 = \mathcal{A}_2^*(W') \cap \mathcal{C}_1(W')$ . Since  $\mathcal{A}_1 \cap \mathcal{C}_1 = \{U\}$ , we have  $|\mathcal{D}_2| \geq m - 2$ . Thus we

may apply Lemma 5.2 to  $W'$ . If Lemma 5.2(i) holds, then  $ge_1 + e_2 + \alpha$  is a term of  $S$  with multiplicity at least  $mn - 1$ , as desired. Therefore Lemma 5.2(ii) holds, whence Lemmas 5.3.3 and 3.2.3 imply that  $a = \pm(f_1 - f_2)$ , contradicting that  $a = -f_2$ . This completes CASE 4.  $\square$

There exists  $e'_2 \in e_2 + nG$  such that  $(e_1, e'_2)$  is a basis for  $G$ . Thus, after changing notation if necessary, we may suppose that  $(e_1, e_2)$  is a basis of  $G$ . If  $g \in G$  and  $x, y \in \mathbb{Z}$  with  $g = xe_1 + ye_2$ , then we set  $\pi_1(g) = xe_1$  and  $\pi_2(g) = ye_2$ .

**CLAIM C:** There exists  $x_0|S_2$  such that  $x - y \in \langle e_1 \rangle$  for all  $xy|x_0^{-1}S_2$ .

*Proof.* We need to show that there exists  $x_0|S_2$  such that  $\pi_2(\psi(x)) = \pi_2(\psi(y))$  for all  $xy|x_0^{-1}S_2$ . We divide the proof into four cases.

CASE 1:  $\Omega_0^u \neq \emptyset$  and there is  $U \in \mathcal{A}_1^* \cap \mathcal{C}_1$ .

In this case, we have

$$(16) \quad ne_1 = \sigma(U) = f_1.$$

Let  $V \in \mathcal{A}_2^*$ . Perform type (II) swaps between  $W_0$  and  $V$ . If  $V, W_0 \in \mathcal{C}_1$ , then we conclude from Lemmas 3.1.1 and 3.4.1 that  $\pi_2(\psi(x)) = \alpha_2$  (say) for all  $x|VW_0^{(2)}$ . If  $V, W_0 \in \mathcal{C}_0$ , then we conclude, from Lemmas 3.1.3 and 3.4.1 and (16), that  $\psi_2$  is constant on  $VW_0^{(2)}$ , whence (16) further implies that  $\pi_2(\psi(x)) = \alpha_2$  for all  $x|VW_0^{(2)}$ . If  $|\{V, W_0\} \cap \mathcal{C}_1| = 1$ , then we conclude from Lemmas 3.1.2 and 3.4.3 that  $\pi_2(\psi(x)) = \alpha_2$  for all  $z|x_0^{-1}VW_0^{(2)}$ , for some  $x_0|VW_0^{(2)}$ . If  $\pi_2(\psi(x_0)) \neq \alpha_2$  and  $x_0|V$ , then pull  $x_0$  up into a new product decomposition  $W'$  and assume we began with  $W'$  instead of  $W$  (note that (16) holds independent of  $W'$  and that  $\tilde{\sigma}(W) = \tilde{\sigma}(W')$  follows by Lemma 3.1.2, so all previous arguments can be applied to  $W'$  regardless of whether  $\mathcal{A}_1^*(W') \cap \mathcal{C}_1(W')$  is nonempty or not). Doing this for all  $V \in \mathcal{A}_2^*$ , we conclude that there is an  $x_0|S_2$  such that  $\pi_2(\psi(x)) = \alpha_2$  for all  $x|x_0^{-1}S_2$ , completing CASE 1.

CASE 2:  $\Omega_0^u \neq \emptyset$  and  $\mathcal{A}_1 \cap \mathcal{C}_1 = \{W_0\}$ .

Performing type II swaps between  $W_0$  and each  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$ , we conclude from Lemmas 3.4.1 and 3.1.1 that  $\pi_2(\psi(x)) = \alpha_2$  (say) for all  $x|W_0^{(2)}U$ , with  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$ . Let w.l.o.g.  $W_1, \dots, W_l$  be the blocks in  $\mathcal{A}_2 \cap \mathcal{C}_0$ , and let  $W_{m+1}, \dots, W_{2m-2}$  be the blocks in  $\mathcal{A}_2^* \cap \mathcal{C}_1$ . Note  $l \geq 1$  else CLAIM C follows by the previous conclusion. Performing type II swaps between  $W_0$  and  $W_j$ , with  $j \in [1, l]$ , we conclude from Lemmas 3.4.3 and 3.1.2 that  $\pi_2(\psi(x)) = \alpha_2$  for all  $x|z_j^{-1}W_j$ , for some  $z_j|W_j$ . We may w.l.o.g. assume  $\pi_2(\psi(z_j)) \neq \alpha_2$  for  $j \in [1, l']$  and  $\pi_2(\psi(z_j)) = \alpha_2$  for  $j \in [l' + 1, l]$ . We have  $l' \geq 2$  else CLAIM C follows.

Perform a type II swap between  $z_1|W_1$  and any term  $y|W_0^{(2)}$ , and let  $W'$  denote the resulting product decomposition. Since  $\pi_2(\psi(z_1)) \neq \alpha_2$ , we are assured that  $\pi_2(\sigma(W_0)) \neq \pi_2(\sigma(W'_0))$ , and hence  $\sigma(W_0) \neq \sigma(W'_0)$ . Thus Lemma 3.1.2 implies that  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$ ,  $W'_0 \in \mathcal{C}_0$  and  $W'_1 \in \mathcal{C}_1$ .

Now pull the term  $z_2|W_2$  up into a new product decomposition  $W''$ . Note by Lemma 3.1.2 that  $\tilde{\sigma}(W'') = \tilde{\sigma}(W)$ . If  $W''_0 \in \mathcal{C}_1$ , then the arguments of the first paragraph show that  $\pi_2(\psi(z_2)) = \alpha_2$ , contradicting that  $l' \geq 2$ . Therefore  $W''_2 \in \mathcal{C}_1$  instead. However, noting that  $yW_0^{(1)}|W''_0$ , for some  $y|W_0^{(2)}$  (since  $\sigma(\iota(W_0^{(2)})) \equiv 1 \pmod{n}$  and  $\sigma(\iota(W'_2)) \equiv 0 \pmod{n}$ ), we can still perform the swap between  $y|W''_0$  and  $z_1|W'_1 = W_1$  described in the previous paragraph, which results in a new product decomposition  $W'''$  in which the  $m$  blocks

$$W'''_1 = W'_1, W'''_2 = W''_2, W'''_{m+1} = W_{m+1}, \dots, W'''_{2m-2} = W_{2m-2}$$

all have equal sum  $f_1$ , contradicting that  $S' \in \mathcal{A}(G)$ , and completing CASE 2.

CASE 3: Either  $(\Omega_0^u \neq \emptyset \text{ and } \mathcal{A}_1 \cap \mathcal{C}_1 = \emptyset)$  or  $(\Omega_0^u = \emptyset \text{ and } W_0 \notin \mathcal{C}_0)$ .

If  $\Omega_0^u = \emptyset$ , we may w.l.o.g. assume  $\tilde{\sigma}(W) = f_1^{m-1} f_2^{m-1} (f_1 + f_2)$  with  $\mathcal{C}_1$  those blocks with sum  $f_1$  and  $\mathcal{C}_2$  those blocks with sum  $f_2$ , and that  $W_0 \in \mathcal{C}_2$ . Let w.l.o.g.  $W_1, \dots, W_s$  be the  $s \leq m-1$  blocks of  $\mathcal{C}_1 \cap \mathcal{A}_2^*$ . Let  $\sigma(W_0) = C f_1 + f_2$  and  $F = (C-1)f_1 + f_2$ . If  $\Omega_0^u \neq \emptyset$ , then we have  $s = m-1$  by hypothesis. If  $s = 0$ , then  $|\mathcal{A}_1^* \cap \mathcal{C}_1| = m-1$ , implying  $e_1$  is a term with multiplicity at least  $mn-1$  in  $S$  (in view of CLAIM B), as desired. Therefore we may assume  $s > 0$ .

We claim, for any  $W$  satisfying the hypothesis of CASE 3 and notated as above (and in fact, if  $W \in \Omega_0^{nu}$ , we will not need that  $\Omega_0^u = \emptyset$ ), that

$$(17) \quad \pi_2(\psi(x_0^{-1} W_0^{(2)} \prod_{\nu=1}^s W_\nu)) = q_2^{(s+1)n-1}$$

for some  $x_0 | W_0^{(2)} \prod_{\nu=1}^s W_\nu$  and  $q_2 \in \text{Ker}(\varphi)$ . To show this, perform type II swaps between  $W_0$  and  $W_i$ ,  $i \in [1, s]$ . If  $\pi_2(F) = 0$ , then Lemmas 3.4.1 and either 3.1.2 or 3.2.3 imply that (17) holds with  $\pi_2(x_0) = q_2$  as well. If  $\pi_2(F) \neq 0$  and (17) fails, then Lemmas 3.4.3 and either 3.1.2 or 3.2.3 imply that  $\pi_2(\psi(z)) = q_2$  (say) for all  $z | x_i^{-1} x_0^{-1} W_0^{(2)} W_i$ , for some  $x_i | W_i$ ,  $i \in [1, s]$ ; moreover,  $s \geq 2$  and w.l.o.g.  $\pi_2(\psi(x_1))$  and  $\pi_2(\psi(x_2))$  are not equal to  $q_2$ . Pull  $x_1 | W_1$  up into a new product decomposition  $W'$ . If  $\sigma(W'_0) = \sigma(W_0)$ , then the arguments of the previous sentence imply either  $\pi_2(\psi(x_1)) = q_2$  or  $\pi_2(\psi(x_2)) = q_2$ , a contradiction. If  $\sigma(W'_0) \neq \sigma(W_0)$  and  $W \in \Omega_0^u$ , then Lemma 3.1.2 implies that  $W' \in \Omega_0^u$  with  $W'_0 \in \mathcal{C}_1$ , whence CLAIM C follows in view of CASE 2 applied to  $W'$ . Therefore we may assume  $\sigma(W'_0) \neq \sigma(W_0)$ ,  $W \in \Omega_0^{nu}$  and  $W'_0 \in \mathcal{C}_1$  (in view of Lemma 3.2.3). Let  $y$  be a term that divides both  $W_0^{(2)}$  and  $W'_0$  (possible since  $\sigma(\iota(W_0)) \equiv 1 \pmod{n}$ ). Choose  $I$  such that  $\min I \equiv \iota(y) \pmod{n}$ , and consequently  $\epsilon(y, z) = 0$  for any  $z$  (in view of (10)). Note that while the new choice of  $I$  may change the overall value of  $\psi(x)$ , where  $x | S_2$ , in a nontrivial manner, nonetheless, the value of  $\pi_2(\psi(x))$  remains unchanged. Perform type II swaps between  $y | W_0$  and any  $z | W_2$ . In view of our choice of  $I$ , Lemma 3.2.3 and  $\pi_2(\psi(x_2)) \neq q_2$ , we conclude that  $-\psi(x_2) + \psi(y) = F = -f_1 + f_2$  (since  $-\pi_2(\psi(x_2)) + \pi_2(\psi(y)) \neq 0$ , implying  $-\psi(x_2) + \psi(y) \neq 0$ ), and that  $-\psi(z) + \psi(y) = 0$  if  $z \neq x_2$  (since  $-\pi_2(\psi(z)) + \pi_2(\psi(y)) = 0$ ); in particular,  $\psi_1(x_2) \neq \psi_1(z)$  for  $z | x_2^{-1} W_2$ . However, performing type II swaps between  $y | W'_0$  and any  $z | W'_2 = W_2$ , we conclude from Lemma 3.2.1 and the choice of  $I$  that  $\psi_1$  is constant on  $W'_2 = W_2$ , contradicting the previous sentence. Thus (17) is established in all cases.

Next we proceed to show that  $s = m-1$ . To this end, suppose  $s < m-1$ . As noted before, we may then assume  $\Omega_0^u = \emptyset$ . Let  $U \in \mathcal{A}_1^* \cap \mathcal{C}_1$  (which is nonempty by the assumption  $s < m-1$ ). Then  $f_1 = \sigma(U) = ne_1$ . Let  $x_0$  and  $q_2$  be as defined by (17). Thus, performing type II swaps between a fixed  $x_1 | x_0^{-1} W_0^{(2)}$  and any  $y | V \in \mathcal{A}_2^* \cap (\mathcal{C}_2 \cup \mathcal{C}_0)$ , we conclude from  $f_1 = \sigma(U) = ne_1$  and Lemmas 3.2.2 and 3.2.5 that  $\psi_2(V) = \psi_2(x_1)^n$  for all such blocks  $V \in \mathcal{A}_1^* \cap (\mathcal{C}_2 \cup \mathcal{C}_0)$ . Hence, in view of  $ne_1 = f_1$ , we conclude that  $\pi_2(\psi(V)) = \pi_2(\psi(x_1))^n = q_2^n$  for all such  $V$ , which combined with (17) implies CLAIM C. So we may assume  $s = m-1$ .

In case  $W \in \Omega_0^{nu}$ , we have assumed  $\Omega_0^u = \emptyset$ . However, we will temporarily drop this assumption, allowing consideration of  $W \in \Omega_0^{nu}$  even when  $\Omega_0^u \neq \emptyset$ , provided it still satisfies the hypothesis of CASE 3 and follows the notation given in the first paragraph with  $s = m-1$ . This will extend until the end of assertion **A1** below, which shows that the exceptional term  $x_0$  in (17) is not necessary.

**A1.** For every  $W \in \Omega_0$  satisfying the hypotheses of CASE 3 (allowing  $W \in \Omega_0^{nu}$  even if  $\Omega_0^u \neq \emptyset$ ), we have  $\pi_2(\psi(x_0)) = q_2$ , where  $q_2$  and  $x_0$  are as given by (17).

*Proof of A1.* Assume instead there exists  $W \in \Omega_0$  satisfying the hypotheses of CASE 3 with  $\pi_2(\psi(x_0)) \neq q_2$ .

Suppose  $x_0|W_j$  with  $j > 0$ . Pull up an arbitrary  $y|W_k \in \mathcal{A}_2$ , with  $k \geq m$ , into a resulting product decomposition  $W''$  (such a block exists, else (17) completes CLAIM C). If  $W''$  satisfies the hypotheses of CASE 3, then applying (17) to  $W''$  we conclude that  $\pi_2(\psi(y)) = q_2$ , whence CLAIM C follows in view of (17) and the arbitrariness of  $y$ . Therefore we may instead assume  $W''$  does not satisfy the hypotheses of CASE 3, whence, in view of CASES 1 and 2, we may assume  $W'' \in \Omega_0^{nu}$  with  $W_0'' \in \mathcal{C}_0(W'')$ .

Let  $z$  be a term dividing both  $W_0^{(2)}$  and  $W_0''^{(2)}$  (which exists in view of  $\sigma(\iota(W_0^{(2)})) \equiv 1 \pmod{n}$ ). Note that we cannot have  $0 = \psi(z) - \psi(x_0) + \epsilon(z, x_0)ne_1$ , as then  $0 = \pi_2(\psi(x_0)) - \pi_2(\psi(z)) = \pi_2(\psi(x_0)) - q_2$ , a contradiction to  $\pi_2(\psi(x_0)) \neq q_2$ . Thus, in view of (17) and Lemma 3.2.3 or 3.1.2, it follows that performing a type II swap between  $x_0|W_j$  and  $z|W_0^{(2)}$  results in a new product decomposition  $W'$  in which  $\sigma(W_j') = Cf_1 + f_2$  and  $\sigma(W_0') = f_1$ . Thus, if  $W \in \Omega_0^u$ , then we can apply Lemma 5.1 to conclude  $W'' \in \Omega_0^u$ , contrary to the conclusion of the previous paragraph. Therefore we may assume  $W \in \Omega_0^{nu}$ . Hence, from  $W'' \in \Omega_0^{nu}$  and Lemma 3.3, it follows that  $\tilde{\sigma}(W'') = \tilde{\sigma}(W)$ , whence  $\sigma(W_0'') = f_1 + f_2$  (in view of  $W_0'' \in \mathcal{C}_0(W'')$ ). However, since  $z|W_0''^{(2)}$ , we may still apply the previously described swap between  $x_0|W_j'' = W_j$  and  $z|W_0''$  now in  $W''$ , which results in a product decomposition  $W''' \in \Omega'$  with  $\nu_{f_2}(\tilde{\sigma}(W''')) = m$  (as  $\sigma(W_j''') = \sigma(W_j') = Cf_1 + f_2 = f_2$  and  $\sigma(W_j'') = \sigma(W_j) = f_1$ ), contradicting that  $S \in \mathcal{A}(G)$ . So we may assume  $x_0|W_0$ .

Perform a type II swap between an arbitrary  $x|W_0^{(2)}$  and  $y|W_j$  with  $j \in [1, m-1]$ . In view of Lemma 3.1.2 or 3.2.3, it follows that

$$(18) \quad \epsilon(x, y)ne_1 + \psi(x) - \psi(y) \in \{0, F\}.$$

If  $x = x_0$ , then it follows, in view of  $\pi_2(\psi(x_0)) - \pi_2(\psi(y)) = \pi_2(\psi(x_0)) - q_2 \neq 0$  and (18), that  $\epsilon(x_0, y)ne_1 + \psi(x_0) - \psi(y) = F$ , and thus

$$(19) \quad 0 \neq \pi_2(\psi(x_0)) - q_2 = \pi_2(\psi(x_0)) - \pi_2(\psi(y)) = \pi_2(F).$$

Consequently, if  $x \neq x_0$ , then, from  $\pi_2(\psi(x)) - \pi_2(\psi(y)) = q_2 - q_2 = 0$  (in view of (17)) and (18) and (19), it follows that

$$\epsilon(x, y)ne_1 + \psi(x) - \psi(y) = 0.$$

As  $y|W_j$  with  $j \in [1, m-1]$  and  $x|x_0^{-1}W_0^{(2)}$  were arbitrary above, we see that we can apply Lemma 5.4 with  $i = 0$ ,  $Z = x_0^{-1}W_0^{(2)}$  and  $\mathcal{D} = \{W_1, \dots, W_{m-1}\}$ .

Thus we can choose  $I$  appropriately so that, for some  $q \in \text{Ker}(\varphi)$ , we have that

$$(20) \quad \psi(x) = q$$

for all  $x|x_0^{-1}W_0^{(2)} \prod_{\nu=1}^{m-1} W_\nu$ , and that

$$(21) \quad \iota(x) \leq \iota(y)$$

for all  $x|x_0^{-1}W_0^{(2)}$  and  $y|W_i$ ,  $i \in [1, m-1]$ . By performing a type II swap between  $x_0|W_0$  and each  $y|W_i$ , with  $i \in [1, m-1]$ , we conclude, from  $\pi_2(\psi(x_0)) \neq q_2 = \pi_2(q)$  and either Lemma 3.1.2 or 3.2.3, that

$$(22) \quad \psi(x_0) - q + \epsilon(x_0, y)ne_1 = (C-1)f_1 + f_2.$$

Thus  $\epsilon(x_0, y)$  must be the same for every  $y|W_j$  with  $j \in [1, m-1]$ . As a result, it follows in view of (10) that either  $\iota(x_0) \leq \min(\text{supp}(\iota(\prod_{\nu=1}^{m-1} W_\nu)))$  or  $\iota(x_0) > \max(\text{supp}(\iota(\prod_{\nu=1}^{m-1} W_\nu)))$ . In the latter case, we may choose  $I$  such  $\min I \equiv \iota(x_0) \pmod n$ , and thus, in both cases, we have (in view of (21))

$$(23) \quad \iota(x) \leq \iota(y)$$

for all  $x|W_0^{(2)}$  and  $y|W_i$ ,  $i \in [1, m-1]$ , while still preserving that (20) holds for some  $q \in \text{Ker}(\varphi)$  (since (23) was all that was required in the proof of Lemma 5.4 to ensure (20) held). Consequently, (22) and (10) imply that

$$(24) \quad \psi(x_0) = q + F = q + (C-1)f_1 + f_2.$$

Let  $y|W_k \in \mathcal{A}_2$  with  $k \geq m$  and  $\pi_2(\psi(y)) \neq q_2$ ; such a term and block exists else CLAIM C follows in view of (17). If  $y|W_k$  could be pulled up into a new product decomposition  $W'$  with  $x_0|W'_0$ , then  $W'$  must still satisfy the hypothesis of CASE 3 (by the same arguments used when  $x_0|W_j$  with  $j > 0$ ), whence applying (17) to  $W'$  implies  $\pi_2(\psi(x_0)) = q_2$  or  $\pi_2(\psi(y)) = q_2$ , contrary to our assumption. Therefore we may assume this is not the case, whence Theorem 2.6.2 implies that

$$(25) \quad \iota(W_0^{(2)}) = g_1^l g_2^{n-1-l} \iota(x_0) \text{ and } \iota(W_k) = g_1^{n-1-l} g_2^l \iota(y),$$

for some  $g_1, g_2 \in \mathbb{Z}$  with  $\gcd(g_1 - g_2, n) = 1$ . If there existed  $x'_0|W_0^{(2)}$  such that  $\epsilon(x'_0, z) = \epsilon(x_0, z)$  for some  $z|W_k$ , then we could apply a type II swap between  $z|W_k$  and each of  $x_0|W_0$  and  $x'_0|W_0$ , which in view of Lemma 3.1.3 or Lemma 3.2 would imply that  $\psi_2(x_0) = \psi_2(x'_0) = \psi_2(q)$ , contradicting (24). Therefore we may assume otherwise, whence (10) implies either

$$(26) \quad \iota(x_0) \leq \min(\text{supp}(\iota(W_k))) \leq \max(\text{supp}(\iota(W_k))) < \min(\text{supp}(\iota(x_0^{-1}W_0^{(2)})))$$

or

$$(27) \quad \iota(x_0) > \max(\text{supp}(\iota(W_k))) \geq \min(\text{supp}(\iota(W_k))) \geq \max(\text{supp}(\iota(x_0^{-1}W_0^{(2)}))).$$

In either case, we see that  $|\text{supp}(\iota(W_k)) \cap \text{supp}(\iota(W_0^{(2)}))| \leq 1$ . As a result, (25) implies that w.l.o.g.  $l = n-1$ ,  $\iota(W_0^{(2)}) = g_1^{n-1} \iota(x_0)$  and  $\iota(W_k) = g_2^{n-1} \iota(y)$ . Thus  $\sigma(\iota(W_k)) \equiv 0 \pmod n$  and  $\sigma(\iota(W_0^{(2)})) \equiv 1 \pmod n$  imply that  $\iota(W_k) = g_2^n$  and  $\iota(x_0) \equiv g_1 + 1 \pmod n$ .

If (26) holds, then from  $\iota(x_0) \equiv g_1 + 1 \pmod n$  and (26) it follows that  $\max I \equiv g_1 \pmod n$ . However, in view of (23), this is only possible if  $\iota(x) \equiv g_1 \pmod n$  for all  $x|x_0^{-1}W_0^{(2)} \prod_{\nu=1}^{m-1} W_\nu$ , in which case, since  $\psi(x) = q$  also holds for all such terms (in view of (20)), it follows that  $S$  contains a term with multiplicity  $mn-1$ , as desired. Therefore we can instead assume (27) holds. In this case, it follows, in view of (27),  $\iota(x_0^{-1}W_0^{(2)}) = g_1^{n-1}$  and  $\iota(x_0) \equiv g_1 + 1 \pmod n$ , that

$$\{g_2\} = \text{supp}(\iota(W_k)) = \text{supp}(\iota(x_0^{-1}W_0^{(2)})) = \{g_1\},$$

contradicting that  $\gcd(g_1 - g_2, n) = 1$ .  $\square$

We now return to arguments where we assume  $\Omega_0^u = \emptyset$  when  $W \in \Omega_0^{nu}$ . In view of **A1**, we may assume  $\pi_2(\psi(x)) = q_2$  for all  $x|W_0^{(2)} \prod_{\nu=1}^{m-1} W_\nu$ . Let  $y|W_k$ , with  $W_k \in \mathcal{A}_2$  and  $k \geq m$ , be arbitrary. If we can pull up  $y$  into a new product decomposition  $W'$  such that either  $W' \in \Omega_0^u$ , or else  $W' \in \Omega_0^{nu}$  and  $W'_0 \notin \mathcal{C}_0(W')$ , then it follows, in view of CASES 1 and 2, **A1** and (17), that we may assume  $\pi_2(\psi(y)) = q_2$  also (note

this is where we need that  $W \in \Omega_0^{nu}$  is allowed in **A1** even when  $\Omega_0^u \neq \emptyset$ ). However, this can only fail if (by an appropriate choice for  $f_2$  in the case when  $W \in \Omega_0^u$ ) w.l.o.g.

$$(28) \quad \tilde{\sigma}(W) = f_1^{m-1} f_2^{m-2} (Cf_1 + f_2)((1-C)f_1 + f_2),$$

with  $\sigma(W_k) = (1-C)f_1 + f_2$  and (recall)  $\sigma(W_0) = Cf_1 + f_2$ . Consequently, we see that there is at most one block  $W_k$  for which this can fail (as  $W_0 \notin \mathcal{C}_0$  when  $\Omega_0^u = \emptyset$ ). As CLAIM C follows otherwise, we may assume  $W_k \in \mathcal{A}_2$  exists and that  $\tilde{\sigma}(W)$  is of such form, and w.l.o.g. assume  $k = 2m - 2$ . Then

$$(29) \quad Cf_1 + f_2 = \sigma(W_0) = Y_1 ne_1 + ne_2 + nq_2,$$

$$(30) \quad f_1 = \sigma(W_1) = Y_2 ne_1 + ne_2 + nq_2,$$

for some  $Y_i \in \mathbb{Z}$ . From (29) and (30), we conclude that

$$(31) \quad (C-1)f_1 + f_2 \in \langle ne_1 \rangle.$$

If there exists  $U \in \mathcal{A}_1^*$ , then  $ne_1 = \sigma(U) = f_2$  (in view of (28),  $s = m - 1$  and  $W_k = W_{2m-2} \in \mathcal{A}_2$ ); thus from (31) it follows that  $(C-1)f_1 \in \langle f_2 \rangle$ , which is only possible if  $C \equiv 1 \pmod{m}$ , contradicting that  $W \notin \mathcal{C}_0$  when  $W \in \Omega_0^{nu}$  (in view of (28)). So we may instead assume  $|\mathcal{A}_1| = 1$ . This same argument also shows that  $\psi_1(ne_1) \neq 0$ . Let  $\mathcal{D} = \{W_m, \dots, W_{2m-2}\}$ .

If  $\psi_2(ne_1) = 0$ , then  $ne_1 \in \langle f_1 \rangle$ , which combined with (31) yields a contradiction to  $(f_1, f_2)$  being a basis. Therefore  $\psi_2(ne_1) \neq 0$ . Thus, in view of Lemma 3.1.3 or Lemmas 3.2.5 and 3.2.2, it follows that we may apply Lemma 5.4 with  $Z = W_0^{(2)}$ ,  $i = 2$  and  $\mathcal{D}$  as given above. Choose  $I$  as directed by Lemma 5.4 (as mentioned before, changing  $I$  does not affect the value of  $\pi_2(\psi(x))$ , and thus (17) remains unaffected). Then

$$(32) \quad \psi_2(x) = \alpha_2,$$

for all  $x | W_0^{(2)} \prod_{\nu=m}^{2m-2} W_\nu$  and some  $\alpha_2 \in \langle f_2 \rangle$ , and

$$(33) \quad \iota(x) \leq \iota(y),$$

for all  $x | W_0^{(2)}$  and  $y | \prod_{\nu=m}^{2m-2} W_\nu$ .

Let  $y_0 | W_{2m-2}$  with  $\pi_2(\psi(y_0)) \neq q_2$  (such  $y_0$  exists, as discussed above, else CLAIM C follows). Let  $W'$  be an arbitrary product decomposition resulting from pulling up  $y_0$  into a new product decomposition. Since  $\pi_2(\psi(y_0)) \neq q_2$ , we have (as discussed earlier)  $\tilde{\sigma}(W') = f_1^{m-1} f_2^{m-1} (f_1 + f_2)$  with  $\sigma(W'_0) = f_1 + f_2$ . Let  $X = \gcd(W_0^{(2)}, W_0'^{(2)})$  and let  $X'$ ,  $Y'$  and  $Y$  be defined by  $W_0^{(2)} = XX'$ ,  $W_0'^{(2)} = XY'$  and  $W_{2m-2} = YY'$ . Thus  $W_{2m-2}' = X'Y$ . Note that all four of these newly defined subsequences are nontrivial in view of  $\sigma(\iota(W_0^{(2)})) \equiv 1 \pmod{n}$  and  $\sigma(\iota(W_{2m-2})) \equiv 0 \pmod{n}$ .

Let  $\mathcal{D}' = \{W_0', W_1', \dots, W_{m-1}'\}$ . In view of Lemma 3.2.4 and  $\psi_1(ne_1) \neq 0$ , it follows that we can apply Lemma 5.4 with  $i = 1$ ,  $Z = W_0'^{(2)}$ , and  $\mathcal{D}$  taken to be  $\mathcal{D}'$  (however, do NOT change  $I$ ). If (9) holds, then (in view of (10)) we can find  $z | W_j'$ , for some  $j \in [1, m-1]$ , such that  $\epsilon(y_0, z) = \epsilon(x, z)$ , where  $x | X$ . Applying a type II swap between  $z | W_j'$  and each of  $x | W_0'$  and  $y_0 | W_0'$ , we conclude from Lemma 3.2.4 that  $\psi_1(x) = \psi_1(y_0)$ . However, since  $x | X$  and  $X | W_0^{(2)}$ , it follows from (32) that  $\psi_2(x) = \psi_2(y_0)$  also, whence  $\psi(x) = \psi(y_0)$ , implying  $q_2 = \pi_2(\psi(x)) = \pi_2(\psi(y_0))$ , contrary to assumption. Therefore we may instead assume (8) holds. Moreover, if both  $y_0$  and some  $x | X$  are contained in the same interval  $J_i$  (from (8)),

then we can repeat the above argument to obtain the same contradiction. Therefore it follows, in view of (33), that  $y_0 \in J_2$  and  $X \subset J_1$ .

Let  $z|W_0'^{(2)}$  and  $z'|W_j'$  with  $j \geq m$  be arbitrary. Performing a type II swap between  $z|W_0'^{(2)}$  and  $z'|W_j'$ , we conclude from Lemma 3.2.5 that

$$\psi_2(z) - \psi_2(z') + \psi_2(\epsilon(z, z')ne_1) = 0.$$

Thus (32) implies that  $\psi_2(\epsilon(z, z')ne_1) = 0$ , which, in view of  $\psi_2(ne_1) \neq 0$  and (10), implies that  $\epsilon(z, z') = 0$  and

$$(34) \quad \iota(z) \leq \iota(z'),$$

for any  $z|W_0'^{(2)}$  and  $z'|W_j'$  with  $j \geq m$ .

Applying (34) using  $z|Y'$  and  $z'|X'$  and  $j = 2m - 2$ , we conclude in view of (33) that

$$(35) \quad \iota(z) = \max(\text{supp}(\iota(W_0'^{(2)}))) = \min(\text{supp}(\iota(\prod_{\nu=m}^{2m-2} W_\nu')))) = \iota(z'),$$

for any  $z'|X'$  and  $z|Y'$ .

From (35) applied with  $z = y_0$ , we see that there is  $y_0'|W_0'^{(2)}$  with  $\iota(y_0') = \iota(y_0)$ . Thus  $y$  can be pulled up into a new decomposition  $W''$  by exchanging  $y_0|W_{2m-2}$  and  $y_0'|W_0$ , and all of the above arguments (valid for an arbitrary  $W'$  obtained by pulling up  $y_0|W_{2m-2}$ ) are applicable for  $W''$ . In particular,  $y_0'^{-1}W_0'^{(2)} = X \subset J_1$  and  $y_0 \in J_2$  imply, in view of  $Y = y_0^{-1}W_{2m-2}$ , (8) and (35), that

$$(36) \quad \max(\text{supp}(\iota(y_0'^{-1}W_0'^{(2)}))) < \min(\text{supp}(\iota(W_{2m-2}))).$$

If we could pull up  $y_0'y_0|W_0W_{2m-2}$  into a new product decomposition  $W'''$ , then (36) would imply that  $X'$  contains a  $z'$  with  $\iota(z') < \iota(y_0)$ , which would contradict (34) applied with  $z = y_0$  and  $z' = z'$ . Therefore we can assume otherwise, whence Theorem 2.6.2 and (36) imply that  $|\text{supp}(\iota(y_0'^{-1}W_0'^{(2)}))| = |\text{supp}(\iota(y_0^{-1}W_{2m-2}))| = 1$ . Thus  $\sigma(\iota(W_0'^{(2)})) \equiv 1 \pmod n$  and  $\sigma(\iota(W_{2m-2})) \equiv 0 \pmod n$  force that  $\iota(W_{2m-2}) = g^n$  and  $\iota(W_0'^{(2)}) = (g-1)^{n-1}g$ , where  $\iota(y_0) = \iota(y_0') = g$ . Consequently, (8),  $X \subset J_1$  and  $y_0 \in J_2$  (in the case when  $W' = W''$ ) force that  $\iota(z) = g$  for all  $z|y_0'^{-1}W_0'^{(2)} \prod_{\nu=1}^{m-1} W_i$ .

Applying type III swaps among the  $W_i$ ,  $i \in [1, m-1]$ , we conclude from Lemma 3.3.1 or 3.1.1 that  $\psi(x) = q$  (say) for all  $x|W_i$ ,  $i \in [1, m-1]$ . Applying type III swaps between  $W_0$  and  $W_1$ , we conclude from Lemma 3.2.3 or 3.1.2 and Lemma 3.4.3 that  $\psi(x) = q$  for all  $x|y_0''^{-1}y_0'^{-1}W_0'^{(2)}$ , for some  $y_0''|y_0'^{-1}W_0'^{(2)}$ , and that  $\psi(y_0'') = q$  or  $q + (C-1)f_1 + f_2$ . Applying a type III swap between  $y_0''|W_0''$  and some  $z|W_1''$  in  $W''$ , we conclude from Lemma 3.2.4 that  $\psi_1(y_0'') = \psi_1(z) = \psi_1(q)$ , whence we see that  $\psi(y_0'') = q + (C-1)f_1 + f_2$  is impossible (since  $C \equiv 1 \pmod m$  would contradict that  $W_0 \notin \mathcal{C}_0$  when  $W \in \Omega_0^{nu}$ ; see (28)). Thus  $\psi(y_0'') = q$  as well, and  $ge_1 + e_2 + q$  has multiplicity at least  $mn - 1$  in  $S$ , as desired, completing CASE 3.

CASE 4:  $\Omega_0^u = \emptyset$  and  $W_0 \in \mathcal{C}_0$ .

We start with the following assertion.

**A2.** If  $\Omega_0^u = \emptyset$ ,  $W \in \Omega_0^{nu}$  with  $\tilde{\sigma}(W) = f_1^{m-1}f_2^{m-1}(f_1 + f_2)$ ,  $W_0 \in \mathcal{C}_0$ , and  $|\mathcal{A}_2 \cap \mathcal{C}_i| \geq 1$  for all  $i \in \{1, 2\}$ , then  $I$  can be chosen such that one of the following properties holds:

- (i)  $|\text{supp}(\psi(W_0'^{(2)}))| = 1$ , or
- (ii) (a)  $\psi_i(ne_1) \neq 0$  for all  $i \in \{1, 2\}$ ,



- (b) there exist  $g_1, g_2 \in \mathbb{Z}$  such that  $\gcd(g_1 - g_2, n) = 1$  and  $\iota(U) = g_1^n$  and  $\iota(V) = g_2^n$ , for every  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$  and  $V \in \mathcal{A}_2^* \cap \mathcal{C}_2$ ,
- (c)  $g_1 > g_2$  and  $\iota(x) \leq g_1$  for all  $x|W_0^{(2)}$ , and
- (d) if also  $|\mathcal{A}_2 \cap \mathcal{C}_i| \geq 2$  for all  $i \in \{1, 2\}$ , then there exist  $c, d \in \text{Ker}(\varphi)$  such that  $\psi(U) = c^n$  and  $\psi(V) = d^n$  for every  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$  and  $V \in \mathcal{A}_2^* \cap \mathcal{C}_2$ .

*Proof of A2.* We may w.l.o.g. assume  $\mathcal{C}_1$  are those blocks with sum  $f_1$ . Performing type II swaps between each  $x|W_0^{(2)}$  and each  $y|U \in \mathcal{A}_2^* \cap \mathcal{C}_1$ , and between each  $x|W_0^{(2)}$  and each  $z|V \in \mathcal{A}_2^* \cap \mathcal{C}_2$ , we conclude from Lemma 3.2 that

$$(37) \quad \psi_1(x) = \psi_1(y) - \psi_1(\epsilon(x, y)ne_1),$$

$$(38) \quad \psi_2(x) = \psi_2(z) - \psi_2(\epsilon(x, z)ne_1),$$

where (10) holds.

Since  $\text{ord}(e_1) = mn$ , one of  $\psi_1(ne_1)$  or  $\psi_2(ne_1)$  is nonzero, say the former (the other case will be identical). Then, in view of (37), we may apply Lemma 5.4 with  $i = 1$ ,  $Z = W_0^{(2)}$  and  $\mathcal{D} = \mathcal{A}_2^* \cap \mathcal{C}_1$ . Consequently, we can choose  $I$  such that

$$(39) \quad \iota(x) \leq \iota(y),$$

for all  $x|W_0^{(2)}$  and  $y|U \in \mathcal{A}_2^* \cap \mathcal{C}_1$ , and  $\psi_1$  is constant on  $W_0^{(2)}$ . If  $\psi_2(ne_1)$  is zero, then (38) implies that  $\psi_2$  is also constant on  $W_0^{(2)}$ , whence (i) holds. Therefore we may assume otherwise, and (a) is established. Likewise, if there is some  $z|V \in \mathcal{A}_2^* \cap \mathcal{C}_2$  with  $\iota(z) \geq \max(\text{supp}(\iota(W_0^{(2)})))$  or  $\iota(z) < \min(\text{supp}(\iota(W_0^{(2)})))$ , then (i) again holds (in view of (10) and (38)). So we may assume otherwise:

$$(40) \quad \min(\text{supp}(\iota(W_0^{(2)}))) \leq \iota(z) < \max(\text{supp}(\iota(W_0^{(2)}))),$$

for all  $z|V \in \mathcal{A}_2^* \cap \mathcal{C}_2$ . Consequently, it follows in view of (39) that both  $\text{supp}(\iota(\prod_{U \in \mathcal{A}_2^* \cap \mathcal{C}_1} U))$  and  $\text{supp}(\iota(\prod_{V \in \mathcal{A}_2^* \cap \mathcal{C}_2} V))$  are disjoint.

Suppose  $|\text{supp}(\iota(U))| > 1$  or  $|\text{supp}(\iota(V))| > 1$ , for some  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$  or  $V \in \mathcal{A}_2^* \cap \mathcal{C}_2$ . Then we may find  $u_0|U$  and  $v_0|V$  such that  $|\text{supp}(\iota(u_0^{-1}U))| > 1$  or  $|\text{supp}(\iota(v_0^{-1}V))| > 1$ , whence it follows, in view of Theorem 2.6.2 (applied to  $\iota(u_0^{-1}v_0^{-1}UV)$  modulo  $n$ ) and the fact that  $\text{supp}(\iota(\prod_{U \in \mathcal{A}_2^* \cap \mathcal{C}_1} U))$  and  $\text{supp}(\iota(\prod_{V \in \mathcal{A}_2^* \cap \mathcal{C}_2} V))$  are disjoint, that we can refactor  $UV = U'V'$  such that  $U'$  and  $V'$  both contain terms from both  $U$  and  $V$ . Replacing the blocks  $U$  and  $V$  by the blocks  $U'$  and  $V'$  yields a new product decomposition  $W' \in \Omega_0$ ; in view of Lemma 3.2.3, we still have  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$ , whence  $W'$  satisfies the hypotheses of **A2**. However, since both  $U'$  and  $V'$  contain terms from both  $U$  and  $V$ , it follows that both  $U'$  and  $V'$  contain a term  $z'|U$  with  $\iota(z') \geq \max(\text{supp}(\iota(W_0^{(2)})))$  (in view of (39)), as well as a term  $z|V$  with  $\min(\text{supp}(\iota(W_0^{(2)}))) \leq \iota(z') < \max(\text{supp}(\iota(W_0^{(2)})))$  (in view of (40)), which makes it impossible for (8) or (9) to hold for  $W'$ , contradicting that the above arguments show Lemma 5.4 must hold for  $W'$ . So we may assume  $|\text{supp}(\iota(U))| = 1$  and  $|\text{supp}(\iota(V))| = 1$  for all  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$  and  $V \in \mathcal{A}_2^* \cap \mathcal{C}_2$ . Moreover, this argument also shows that if  $\iota(U) = g_1^n$  and  $\iota(V) = g_2^n$ , then  $\gcd(g_1 - g_2, n) = 1$ .

Suppose  $|\text{supp}(\iota(\prod_{U \in \mathcal{A}_2^* \cap \mathcal{C}_1} U))| > 1$  or  $|\text{supp}(\iota(\prod_{V \in \mathcal{A}_2^* \cap \mathcal{C}_2} V))| > 1$ , say the former (the other case will be identical). Then there are  $U_1, U_2 \in \mathcal{A}_2^* \cap \mathcal{C}_1$  and  $V \in \mathcal{A}_2^* \cap \mathcal{C}_2$  with  $\iota(U_1) = g_1$ ,  $\iota(U_2) = g'_1$  and  $\iota(V) = g_2$ , where  $g_1 \neq g'_1$ . We have  $\gcd(g_1 - g'_1, n) = 1$ , else repeating the arguments of the previous paragraph, using  $U_1$  and  $U_2$  in place of  $U$  and  $V$ , we obtain a  $W' \in \Omega_0$  satisfying the hypotheses of **A2** but such that the conclusion of the previous paragraph fails, whence  $1 = |\text{supp}(\psi(W_0^{(2)}))| = |\text{supp}(\psi(W_0^{(2)}))|$  must hold

by prior arguments, yielding (i). Hence, since  $\gcd(g_1 - g_2, n) = 1$  and  $\gcd(g'_1 - g_2, n) = 1$ , it follows that all  $n$ -term zero-sum modulo  $n$  subsequences of  $g_1^{n-1}g_1'^{n-1}g_2^{n-1}$  have support of cardinality three. Thus, by two applications of Theorem 2.6.1, we see that we can refactor  $U_1U_2V = XYZ$  such that  $X, Y$  and  $Z$  all contain terms from each of  $U_1, U_2$  and  $V$  (note, since  $|\text{supp}(\iota(X))| = 3$ , that  $\iota(YZ) \subset g_1^{n-1}g_1'^{n-1}g_2^{n-1}$ ). Replacing  $U_1, U_2$  and  $V$  by  $X, Y$  and  $Z$  yields a new product decomposition  $W' \in \Omega_0$ ; in view of  $\Omega_0^u = \emptyset$  and  $m \geq 5$ , we still have  $\tilde{\sigma}(W') = \tilde{\sigma}(W)$ , whence  $W'$  satisfies the hypotheses of **A2**. However, since  $X, Y$  and  $Z$  each contain terms from  $U_1, U_2$  and  $V$ , we see that the condition  $|\text{supp}(\iota(U))| = 1$  for  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$  fails for  $W'$ , whence previous arguments show  $|\text{supp}(\psi(W_0^{(2)}))| = |\text{supp}(\psi(W_0'^{(2)}))| = 1$ , yielding (i). So we may assume  $|\text{supp}(\iota(\prod_{U \in \mathcal{A}_2^* \cap \mathcal{C}_1} U))| = 1$  and  $|\text{supp}(\iota(\prod_{V \in \mathcal{A}_2^* \cap \mathcal{C}_2} V))| = 1$ , and w.l.o.g. assume  $\text{supp}(\iota(\prod_{U \in \mathcal{A}_2^* \cap \mathcal{C}_1} U)) = g_1$  and  $\text{supp}(\iota(\prod_{V \in \mathcal{A}_2^* \cap \mathcal{C}_2} V)) = g_2$ . This establishes (b). Moreover, by the arguments from the second paragraph, we see that we can choose  $I$  such that (c) holds.

We now assume  $|\mathcal{A}_2 \cap \mathcal{C}_i| \geq 2$ , for all  $i \in \{1, 2\}$ . Performing type III swaps between distinct  $U_1, U_2 \in \mathcal{A}_2^* \cap \mathcal{C}_1$  and between distinct  $V_1, V_2 \in \mathcal{A}_2^* \cap \mathcal{C}_2$ , we conclude from Lemma 3.3 that  $\psi(U) = c$  (say) for all  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$  and that  $\psi(U) = d$  (say) for all  $V \in \mathcal{A}_2^* \cap \mathcal{C}_2$ , establishing (d), and completing the proof of **A2**.  $\square$

Since  $\Omega_0^u = \emptyset$ , it follows, in view of Lemma 3.3, that if we pull up any term  $y|U$ , where  $U \in \mathcal{A}_2^*$ , then we may assume the resulting product decomposition still satisfies the hypothesis of CASE 4, else CASE 3 completes the proof. Thus, if for every product decomposition satisfying the hypothesis of CASE 4 we can find  $I$  such that  $|\text{supp}(\psi(W_0^{(2)}))| = 1$ , then, since modifying  $I$  does not alter the values  $\pi_2(\psi(x))$ , we would be able to conclude  $|\text{supp}(\pi_2(\psi(S_2)))| = 1$ —by successively pulling up terms  $y|S_2$ , yielding a sequence of product decompositions satisfying the hypotheses of CASE 4, until every such  $y$  occurred in the  $W_0^{(2)}$  part of one of these product decompositions, and then noting that there must always be a common term in  $W_0^{(2)}$  between any two consecutive product decompositions in the sequence (in view of  $\sigma(\iota(W_0^{(2)})) \equiv 1 \pmod{n}$ )—completing CLAIM C. Therefore we may assume this is not the case for  $W$ . Let w.l.o.g.  $\tilde{\sigma}(W) = f_1^{m-1}f_2^{m-1}(f_1 + f_2)$  and  $\mathcal{C}_1$  consist of those blocks with sum  $f_1$ .

Note that we must have  $\mathcal{A}_2^* \cap \mathcal{C}_1$  and  $\mathcal{A}_2^* \cap \mathcal{C}_2$  both nonempty, else in view of CLAIM B it would follow that  $e_1$  is a term of  $S$  with multiplicity  $mn - 1$ , completing the proof. Thus **A2(ii)(a)** implies that  $\psi_i(ne_1) \neq 0$  for  $i \in \{1, 2\}$ . As a result, we cannot have a block  $U \in \mathcal{A}_1^*$  (else  $ne_1 = \sigma(U) = f_1$  or  $f_2$ ). Hence  $|\mathcal{A}_1| = 1$ , implying  $|\mathcal{A}_2^* \cap \mathcal{C}_1| \geq 2$  and  $|\mathcal{A}_2^* \cap \mathcal{C}_2| \geq 2$ . Thus, by choosing  $I$  appropriately, **A2(ii)(a–d)** holds for  $W$ .

Suppose  $\text{supp}(\iota(W_0^{(2)})) \neq \{g_1, g_2\}$ . Then there must be some  $x_0|W_0^{(2)}$  with  $\iota(x_0) \notin \{g_1, g_2\}$  (in view of  $\sigma(\iota(W_0^{(2)})) \equiv 1 \pmod{n}$ ). Since  $\gcd(g_1 - g_2, n) = 1$ , there is no  $n$ -term zero-sum mod  $n$  subsequence of  $g_1^{n-1}g_2^{n-1}$ . Thus applying Theorem 2.6.1 to  $g_1^{n-1}g_2^{n-1}\iota(x_0)$  implies that we may find a subsequence  $U_1|W_0^{(2)}UV$ , where  $U \in \mathcal{A}_2^* \cap \mathcal{C}_1$  and  $V \in \mathcal{A}_2^* \cap \mathcal{C}_2$ , such that  $x_0|U_1$  and  $\text{supp}(\iota(z^{-1}U_1)) = \{g_1, g_2\}$ . Consequently,  $v_{g_i}(U_1) \leq n - 2$ , and thus  $v_{g_i}(\iota(U_1^{-1}W_0^{(2)}UV)) \geq 2$ , for  $i = \{1, 2\}$ . Thus, if there were no  $n$ -term zero-sum mod  $n$  subsequence of  $\iota(U_1^{-1}u_1^{-1}v_1^{-1}W_0^{(2)}UV)$ , where  $u_1|U_1^{-1}U$  and  $v_1|U_1^{-1}V$ , then Theorem 2.6.2 would imply that  $\iota(U_1^{-1}W_0^{(2)}UV) = g_1^n g_2^n$ , whence

$$1 \equiv \sigma(\iota(W_0^{(2)}UV)) \equiv \sigma(\iota(U_1)) + ng_1 + ng_2 \equiv 0 \pmod{n},$$

which is a contradiction. Therefore we may assume there exists such a subsequence  $\iota(U_2)$ , where  $U_2|U_1^{-1}u_1^{-1}v_1^{-1}W_0^{(2)}UV$ . Let  $W'_0$  be defined by  $W_0UV = U_1U_2W'_0$ . Then replacing the blocks  $W_0, U$

and  $V$  with the blocks  $W'_0$ ,  $U_1$ , and  $U_2$  yields a new product decomposition  $W' \in \Omega_0$ . Since  $\Omega_0^u = \emptyset$  and  $m \geq 4$ , we must have  $\tilde{\sigma}(W) = \tilde{\sigma}(W')$ , and we may further assume  $W'_0 \in \mathcal{C}_0$  else CASE 3 completes the proof. Thus  $W'$  satisfies the hypotheses of CASE 4, but since  $|\text{supp}(\iota(U_1))| > 1$ , we see that  $W'$  does not satisfy **A2.(ii)**. Thus **A2.(i)** implies that we must have  $|\text{supp}(\pi_2(\psi(W'_0{}^{(2)})))| = 1$  (note we do not have  $|\text{supp}(\psi(W'_0{}^{(2)}))| = 1$  as we would need to change  $I$  for this to hold); since  $u_1 v_1 | W'_0$ , this implies that  $\pi_2(c) = \pi_2(\psi(u_1)) = \pi_2(\psi(v_1)) = \pi_2(d)$ .

Let  $x|x_0^{-1}W_0^{(2)}$  be arbitrary. By Theorem 2.6.1, it follows that there is an  $n$ -term zero-sum mod  $n$  subsequence of  $\iota(x^{-1}U_1^{-1}W_0^{(2)}UV)$ , say  $\iota(U_3)$  with  $U_3|x^{-1}U_1^{-1}W_0^{(2)}UV$ . Let  $W''_0$  be defined by  $W_0UV = U_1U_3W''_0$ . Then replacing the blocks  $W_0$ ,  $U$  and  $V$  with the blocks  $W''_0$ ,  $U_1$ , and  $U_3$  yields a new product decomposition  $W'' \in \Omega_0$ , and as before we may assume  $W''$  satisfies the hypotheses of CASE 4 with  $\tilde{\sigma}(W'') = \tilde{\sigma}(W)$ . Thus, since  $|\text{supp}(\iota(U_1))| > 1$ , we see that  $W''$  does not satisfy **A2.(ii)**, and so we must have

$$(41) \quad |\text{supp}(\pi_2(\psi(W''_0{}^{(2)})))| = 1.$$

Since  $x_0|U_1$ , it follows in view of the pigeonhole principle that we must have a term  $x'|W''_0{}^{(2)}$  with  $x'|UV$ , and thus with  $\pi_2(\psi(x')) = \pi_2(c)$  (in view of the previous paragraph). Since  $x|W''_0$ , this implies  $\pi_2(\psi(x)) = \pi_2(c)$  (in view of (41)). As  $x|x_0^{-1}W_0^{(2)}$  was arbitrary, we conclude that every  $x|x_0^{-1}S_2$  has  $\pi_2(\psi(x)) = \pi_2(c) = \pi_2(d)$ , completing the proof (in view of **A2.(ii)** holding for  $W$ ). So we may instead assume  $\text{supp}(\iota(W_0^{(2)})) = \{g_1, g_2\}$ .

Since  $|\mathcal{A}_1| = 1$ , let w.l.o.g.  $W_1, \dots, W_{m-1}$  be the blocks of  $\mathcal{A}_2^* \cap \mathcal{C}_1$ , and let  $W_m, \dots, W_{2m-2}$  be the blocks of  $\mathcal{A}_2^* \cap \mathcal{C}_2$ . Let  $W_0^{(2)} = b_1 \dots b_t b'_1 \dots b'_{n-t}$  with  $\iota(b_i) = g_1$  and  $\iota(b'_j) = g_2$ . Applying type III swaps between  $b_i|W_0$  and  $y|W_1$ , it follows from Lemma 3.3.4 that we may assume  $\psi(b_i) = \psi(y) = c$  for all  $i$  (else CASE 3 completes the proof). Likewise applying type III swaps between  $b'_i|W_0$  and  $z|W_m$ , it follows that  $\psi(b'_i) = \psi(z) = d$  for all  $i$ . Consequently, we may assume  $t \in [2, n-2]$ , else  $S$  contains a term with multiplicity at least  $mn-1$ , as desired (either  $g_1 e_1 + e_2 + c$  or  $g_2 e_1 + e_2 + d$ ).

Applying type II swaps between  $b_1|W_0$  and  $z|W_m$  and between  $b'_1|W_0$  and  $y|W_1$ , it follows, in view of Lemma 3.2, (10) and  $g_1 > g_2$ , that

$$(42) \quad d - c \in \langle f_2 \rangle,$$

$$(43) \quad c - d + ne_1 \in \langle f_1 \rangle.$$

Since  $t \in [2, n-2]$ , we have  $b_1 b_2 | W_0^{(2)}$  and  $b'_1 b'_2 | W_0^{(2)}$ . Let  $Y$  be a subsequence of  $W_1$  and  $Z$  be a subsequence of  $W_m$  with  $|Y| = |Z| = 2$ . Applying type II swaps between  $b'_1 b'_2 | W_0$  and  $Y | W_1$  and between  $b_1 b_2 | W_0$  and  $Z | W_m$ , we conclude from Lemma 3.2 that

$$(44) \quad 2(d - c) + \epsilon(b'_1 b'_2, Y)ne_1 \in \langle f_2 \rangle,$$

$$(45) \quad 2(c - d) + \epsilon(b_1 b_2, Z)ne_1 \in \langle f_1 \rangle.$$

Observe (in view of  $g_1 > g_2$ ) that

$$\epsilon(b'_1 b'_2, Y)ne_1 = \begin{cases} 0, & \text{if } g_1 - g_2 \leq \frac{n-1}{2}; \\ -ne_1, & \text{if } g_1 - g_2 \geq \frac{n+1}{2}. \end{cases}$$

Likewise

$$\epsilon(b_1 b_2, Z) n e_1 = \begin{cases} n e_1, & \text{if } g_1 - g_2 \leq \frac{n-1}{2}; \\ 2n e_1, & \text{if } g_1 - g_2 \geq \frac{n+1}{2}. \end{cases}$$

Thus, if  $g_1 - g_2 \leq \frac{n-1}{2}$ , then (45) and (43) imply that  $c - d \in \langle f_1 \rangle$ , which combined with (42) implies that  $c = d$ , in which case CLAIM C follows. On the other hand, if  $g_1 - g_2 \geq \frac{n+1}{2}$ , then (44) and (42) imply that  $n e_1 \in \langle f_2 \rangle$ , which contradicts that **A2.(ii)(a)** holds for  $W$ , completing CASE 4.  $\square$

**CLAIM D:**  $h(S) = mn - 1$ .

*Proof.* Let  $S'_2 = x_0^{-1} S_2$ , with  $x_0$  as in CLAIM C, and let  $S' = S_1 S'_2$ . By Proposition 4.2 and CLAIM B, we have  $S_1 = e_1^{|S_1|}$ ,  $|S_1| = \ell n - 1$  and  $|S'_2| = 2mn - \ell n - 1$ , for some  $\ell \geq 1$ . If  $\ell \geq m$ , then  $e_1$  is a term with multiplicity at least  $mn - 1$ , as desired. Therefore  $\ell < m$ . Moreover, since  $S \in \mathcal{A}(G)$ , it follows that  $0 \notin \Sigma(S')$ . In view of CLAIM C, we may assume every  $x|S'_2$  is of the form  $y_i e_1 + (1 + nq)e_2$ , with  $q \in [0, m - 1]$ . Let  $T = \pi_1(S'_2) \in \mathcal{F}(\langle e_1 \rangle)$ , and let  $H' = \langle e_1, (1 + qn)e_2 \rangle \cong C_{mn} \oplus C_{rn}$ , where  $rn = \text{ord}((1 + qn)e_2)$ . If  $r < m$ , then noting that  $S' \in \mathcal{F}(H')$  with  $|S'| = 2mn - 2 \geq mn + rn - 1 = D(H')$ , we see that  $0 \in \Sigma(S')$ , contradicting that  $S \in \mathcal{A}(G)$ . Thus we may choose  $e_2$  to be  $(1 + qn)e_2$  while still preserving that  $(e_1, e_2)$  is a basis, and so w.l.o.g. we assume  $q = 0$ .

Since  $\ell < m$ , it follows that  $|S'_2| = 2mn - \ell n - 1 \geq mn + n - 1 \geq mn + 2$  and

$$(46) \quad \Sigma(S_1) = \{e_1, 2e_1, \dots, (\ell n - 1)e_1\}.$$

Consequently,  $0 \notin \Sigma(S')$  implies

$$(47) \quad \Sigma_{mn}(S'_2) = \Sigma_{mn}(T) \subset A := \{e_1, 2e_1, \dots, (mn - \ell n)e_1\},$$

and thus

$$(48) \quad |\Sigma_{mn}(T)| \leq mn - \ell n = |T| - mn + 1.$$

Note  $h(T) = h(S'_2) \leq mn - 2$ , else the proof is complete. Thus we can apply Theorem 2.7, taking  $k = 3$ , whence it follows, in view of (48) and  $0 \notin \Sigma_{mn}(T)$ , that  $|\text{supp}(T)| \leq 2$ .

We may assume  $|\text{supp}(T)| = 2$ , else  $S$  will contain a term with multiplicity  $|T| = 2mn - \ell n - 1 \geq mn + n - 1$ , contradicting that  $S \in \mathcal{A}(G)$ . Thus  $T = (g_0 e_1)^{n_1} ((g_0 + d)e_1)^{n_2}$  for some  $g_0, d \in \mathbb{Z}$  with  $de_1 \neq 0$ . Since  $(e_1, g_0 e_1 + e_2)$  is also a basis for  $G$ , then, by redefining  $e_2$  to be  $g_0 e_1 + e_2$ , we may w.l.o.g. assume  $g_0 = 0$ . Thus

$$(49) \quad \Sigma_{mn}(T) = B := (mn - n_1)de_1 + \{0, de_1, \dots, (mn - \ell n - 1)de_1\},$$

which is an arithmetic progression of difference  $de_1$  and length  $mn - \ell n$  (in view of  $0 \notin \Sigma_{mn}(T)$ ). In view of (47), we have  $B = A$  with

$$2 \leq n \leq |A| = mn - \ell n \leq mn - n \leq mn - 2.$$

Thus  $de_1 = \pm e_1$  (as the difference of an arithmetic progression under the above assumptions is unique up to sign). Consequently, (47) and (49) imply that  $n_1 = nm - 1$  if  $de_1 = e_1$  (since  $|S'| \leq 2nm - 2$ ), and that  $n_1 = mn - \ell n$  if  $de_1 = -e_1$  (since  $|S'| < 2mn - \ell n$ ). However, in the former case,  $e_2$  has the desired multiplicity in  $S$ , while in the latter case,  $n_2 = 2mn - \ell n - 1 - n_1 = mn - 1$ , and thus  $de_1 + e_2 = -e_1 + e_2$  has the desired multiplicity, completing the proof.  $\square$

## 6. PROOF OF THE COROLLARY

Let  $G = C_{n_1} \oplus C_{n_2}$ , with  $1 < n_1 \mid n_2$ , and suppose that, for every prime divisor  $p$  of  $n_1$ , the group  $C_p \oplus C_p$  has Property **B**. The assertion that  $C_{n_1} \oplus C_{n_1}$  has Property **B** follows from the Theorem and from the following two statements:

- (a) For every  $n \in [2, 10]$ , the group  $C_n \oplus C_n$  has Property **B**: for  $n \leq 6$  this may be found in [5, Proposition 4.2]; the cases  $n \in \{8, 9, 10\}$  (and more) are settled in [1].
- (b) If  $n \geq 6$  and  $C_n \oplus C_n$  has Property **B**, then  $C_{2n} \oplus C_{2n}$  has Property **B** (see [5, Theorem 8.1]).

Since  $C_{n_1} \oplus C_{n_1}$  has Property **B**, the characterization of the minimal zero-sum sequences over  $G$  of length  $D(G)$  now follows from [13, Theorem 3.3].  $\square$

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